Chapter 2  Differentiation

Chapter Summary

Section Topics

2.1  **The Derivative and the Tangent Line Problem**—Find the slope of the tangent line to a curve at a point. Use the limit definition to find the derivative of a function. Understand the relationship between differentiability and continuity.

2.2  **Basic Differentiation Rules and Rates of Change**—Find the derivative of a function using the Constant Rule. Find the derivative of a function using the Power Rule. Find the derivative of a function using the Constant Multiple Rule. Find the derivative of a function using the Sum and Difference Rules. Find the derivatives of the sine function and of the cosine function. Use derivatives to find rates of change.

2.3  **The Product and Quotient Rules and Higher-Order Derivatives**—Find the derivative of a function using the Product Rule. Find the derivative of a function using the Quotient Rule. Find the derivative of a trigonometric function. Find a higher-order derivative of a function.

2.4  **The Chain Rule**—Find the derivative of a composite function using the Chain Rule. Find the derivative of a function using the General Power Rule. Simplify the derivative of a function using algebra. Find the derivative of a trigonometric function using the Chain Rule.

2.5  **Implicit Differentiation**—Distinguish between functions written in implicit form and explicit form. Use implicit differentiation to find the derivative of a function.

2.6  **Related Rates**—Find a related rate. Use related rates to solve real-life problems.

Chapter Comments

The material presented in Chapter 2 forms the basis for the remainder of calculus. Much of it needs to be memorized, beginning with the definition of a derivative of a function found on page 99. Your students need to have a thorough understanding of the tangent line problem and they need to be able to find an equation of a tangent line. Frequently, students will use the function \( f'(x) \) as the slope of the tangent line. They need to understand that \( f'(x) \) is the formula for the slope and the actual value of the slope can be found by substituting into \( f'(x) \) the appropriate value for \( x \). On page 102 of Section 2.1, you will find a discussion of situations where the derivative fails to exist. These examples (or similar ones) should be discussed in class.

As you teach this chapter, vary your notations for the derivative. One time write \( y' \), another time write \( dy/dx \) or \( f'(x) \). Terminology is also important: instead of saying “find the derivative,” sometimes say, “differentiate.” This would be an appropriate time, also, to talk a little about Leibnitz and Newton and the discovery of calculus.

Sections 2.2, 2.3, and 2.4 present a number of rules for differentiation. Have your students memorize the Product Rule and the Quotient Rule (Theorems 2.7 and 2.8) in words rather than symbols. Students tend to be lazy when it comes to trigonometry and therefore, you need to impress upon them that the formulas for the derivatives of the six trigonometric functions need to be memorized also. You will probably not have enough time in class to prove every one of these differentiation rules, so choose several to do in class and perhaps assign a few of the other proofs as homework.
The Chain Rule, in Section 2.4, will require two days of your class time. Students need a lot of practice with this and the algebra involved in these problems. Many students can find the derivative of \( f(x) = x^2\sqrt{1-x^2} \) without much trouble, but simplifying the answer is often very difficult for them. Insist that they learn to factor and write the answer without negative exponents. Strive to get the answer in the form given in the back of the book. This will help them later on when the derivative is set equal to zero.

Implicit differentiation is often difficult for students. Have your students think of \( y \) as a function of \( x \) and therefore \( y^3 \) is \( [f(x)]^3 \). This way they can relate implicit differentiation to the Chain Rule studied in the previous section.

Try to get your students to see that Related Rates, discussed in Section 2.6, are another use of the Chain Rule.

Section 2.1 The Derivative and the Tangent Line Problem

Tips and Tools for Problem Solving

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\textbf{Exercises 33–38 (33–36 in Calculus 8/e)}

To give students more practice, we added Exercises 33 and 34. We also suggest doing the following. After going over Example 3, return to Example 2 where \( f(x) = x^2 + 1 \) and note that \( f'(x) = 2x \).

How can we find the equation of the line tangent to \( f \) and parallel to \( 4x - y = 0 \)? Because the slope of the line is 4,

\begin{align*}
2x &= 4 \\
x &= 2.
\end{align*}

So, at the point \((2, 5)\) the tangent line is parallel to \( 4x - y = 0 \). The equation of the tangent line is \( y - 5 = 4(x - 2) \) or \( y = 4x - 3 \).

\textit{Capstone}

\textbf{Page 105, Exercise 64} You can use this exercise to review (or discover) the following concepts.

- Estimating values on a graph
- Interpreting values of the derivative of a function
- Discovering relationships between a function and its derivative

This \textit{Capstone} involves using the derivative of an unknown function. You can use this exercise to introduce concepts such as increasing and decreasing.

Parts (a) and (b) can be answered using the graph of \( g' \). (See the transparency.) You can use parts (c) and (d) to introduce the terms increasing and decreasing (or you can just say rising and falling, terms familiar from our study of slope). When \( g'(x) \) is negative, \( g \) is decreasing (or falling) and when \( g'(x) \) is positive, \( g \) is increasing (or rising). Remind students that we know this because the derivative tells us the slope of the tangent line at each point. Alternatively, you can demonstrate this by graphing \( g(x) = \frac{1}{6}x^3 - 3x \) and noting the behavior of \( g \) at \( x = 1 \) and \( x = -4 \). For part (e), note that \( g'(x) \) is positive on the interval \([4, 6]\). This means that \( g \) is increasing from \( x = 4 \) to \( x = 6 \). So,

\begin{align*}
g(6) &> g(4) \quad \text{and} \quad g(6) - g(4) > 0. \quad (\text{You can demonstrate this numerically using } g(x) = \frac{1}{6}x^3 - 3x) \\
\end{align*}

Finally, it is not possible to find \( g(2) \) from the graph, but you can say that \( g \) is decreasing at \( x = 2 \) using the same reasoning as in part (c).
Solution

(a) \( g'(0) = -3 \)

(b) \( g'(3) = 0 \)

(c) Because \( g'(1) = -\frac{8}{3} \), \( g \) is decreasing (falling) at \( x = 1 \).

(d) Because \( g'(-4) = \frac{7}{3} \), \( g \) is increasing (rising) at \( x = -4 \).

(e) Because \( g'(4) \) and \( g'(6) \) are both positive, \( g(6) \) is greater than \( g(4) \), and \( g(6) - g(4) > 0 \).

(f) No, it is not possible. All you can say is that \( g \) is decreasing (falling) at \( x = 2 \).

Section 2.2 Basic Differentiation Rules and Rates of Change

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Exercises 3–24 (3–24 in Calculus 8/e)
We rewrote the direction line to clarify that the derivatives should be found using rules of differentiation.

Exercises 39–54 (39–52 in Calculus 8/e)
To give students more practice, we added Exercises 43 and 44. Some students need to be reminded that they can rewrite the function before differentiating. We suggest you go over an example in class like \( f(x) = \frac{5x^2 + x}{x} \). Show students that before differentiating they can rewrite the function as \( f(x) = 5x + 1 \). Then they can differentiate to obtain \( f'(x) = 5 \). Use this example to emphasize the prudence of examining the function first before differentiating. Rewriting the function in a simpler, equivalent form can expedite the differentiating process.

Exercises 59–64 (57–62 in Calculus 8/e)
Students had trouble with these exercises. Students may need to review finding the zeros of a polynomial function and solving trigonometric equations. See Sections 2.2 and Section 5.3 of Precalculus, 7th edition, by Larson and Hostetler respectively.

Exercises 65–70 (63–66 in Calculus 8/e)
To give students more practice, we added Exercises 69 and 70.

Capstone

Page 116, Exercise 72 You can use this exercise to review the following concepts.

- Analyzing the graph of a function to determine when the average rate of change of the function is greatest
- Analyzing the graph of a function to compare the average rate of change and instantaneous rate of change of the function
- Sketching the graph of a tangent line so that the slope is the same as the average rate of change over a given interval

Use the graph of \( f \) given on the transparency to reinforce the solution on the next page. A transparency is also provided for part (c).
Solution
(a) The slope appears to be steepest between $A$ and $B$.
(b) The average rate of change between $A$ and $B$ is greater than the instantaneous rate of change at $B$.

Section 2.3  Product and Quotient Rules and Higher-Order Derivatives

Tips and Tools for Problem Solving

We noticed in the data that after applying differentiation rules, some students are having difficulty simplifying polynomial and rational expressions. Students should review these concepts by studying Appendices A.2–A.4 and A.7 in Precalculus, 7th edition, by Larson and Hostetler.

Exercises 73–76 (73–76 in Calculus 8/e)
We revised Exercise 73 and reordered the exercises to improve the grading.

Exercises 93–100 (93–98 in Calculus 8/e)
To give students more practice, we added Exercises 93 and 94.

Capstone

Page 129, Exercise 120  You can use this exercise to review the following concepts.

- Analyzing the graph of a function and the graphs of its first and second derivatives
- Identifying the graphs of position, velocity, and acceleration functions
- Using the graphs of position, velocity, and acceleration functions to identify when a particle speeds up and when it slows down

This exercise will help you to review higher-order derivatives as well as the relationship between position, velocity, and acceleration functions. Using the transparency for part (a), note that the function that looks cubic is the position function, the function that looks quadratic is the velocity function, and the function that is a line is the acceleration function. You can demonstrate this for students using the cubic function $f(x) = x^3$. Note that the first derivative is $f'(x) = x^2$ and the second derivative is $f''(x) = x$. Recall from Section 2.2, page 114, that the speed of an object is the absolute value of its velocity. Show the transparency for part (b), which shows the graph of the absolute value of the velocity. (Note that the graph is obtained by reflecting in the $x$-axis the negative portion of the graph of $v$.) From this graph, show the intervals where the particle speeds up and when it slows down (see the solution on the next page).
Solution

(a) 

\[ s(t) \text{ position function} \\
 v(t) \text{ velocity function} \\
a(t) \text{ acceleration function} \\

(b) The speed of the particle is the absolute value of its velocity. So, the particle’s speed is slowing down on the intervals \((0, 4/3)\) and \((8/3, 4)\) and it speeds up on the intervals \((4/3, 8/3)\) and \((4, 6)\).

Section 2.4 The Chain Rule

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Exercises 7–36 (7–32 in Calculus 8/e)
To give students more practice, we added Exercises 33–36. Students need a lot of practice with the Chain Rule and the algebra involved in these problems. For students with poor algebra skills, have them review Appendix A.7: Errors and the Algebra of Calculus of Precalculus, 7th edition, by Larson and Hostetler.

Exercises 45–66 (41–58 in Calculus 8/e)
To give students more practice, we added Exercises 57, 58, 65, and 66.

Exercises 91–96 (83–86 in Calculus 8/e)
To give students more practice, we added Exercises 91 and 94. We also revised Exercises 92 and 93 and reordered the block of exercises to improve the grading.

Capstone

Page 139, Exercise 108 You can use this exercise to review the following concepts.

- Product Rule
- Chain Rule
- Quotient Rule
- General Power Rule

Students need to understand these rules because they are the foundation of our study of differentiation.

Use the solution to show students how to solve each problem. As you apply each rule, give the definition of the rule verbally. Note that part (b) is not possible because we are not given \( g'(3) \).
Solution

(a) \( f(x) = g(x)h(x) \)
\[ f'(x) = g(x)h'(x) + g'(x)h(x) \]
\[ f'(5) = (-3)(-2) + (6)(3) = 24 \]

(b) \( f(x) = g(h(x)) \)
\[ f'(x) = g'(h(x))h'(x) \]
\[ f'(5) = g'(3)(-2) = -2g'(3) \]
Not possible. You need \( g'(3) \) to find \( f'(5) \).

(c) \( f(x) = \frac{g(x)}{h(x)} \)
\[ f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2} \]
\[ f'(5) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \frac{4}{3} \]

(d) \( f(x) = [g(x)]^3 \)
\[ f'(x) = 3[g(x)]^2 g'(x) \]
\[ f'(5) = 3(-3)^2(6) = 162 \]

Section 2.5 Implicit Differentiation

Tips and Tools for Problem Solving

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Although we made changes to the section exercises, no changes were made based on the data.

Capstone

Page 148, Exercise 74 You can use this exercise to review the following concepts.

- Finding derivatives when the variables agree and when they disagree
- Using implicit differentiation to find the derivative of a function

Implicit differentiation is often difficult for students, so as you review this concept remind students to think of \( y \) as a function of \( x \). Part (a) is true, and part (b) can be corrected as shown below. Part (c) requires implicit differentiation. Note that the result can also be written as \( -2y \sin(y^2) \frac{dy}{dx} \).

Solution

(a) True

(b) False. \( \frac{d}{dy} \cos(y^2) = -2y \sin(y^2) \).

(c) False. \( \frac{d}{dx} \cos(y^2) = -2yy' \sin(y^2) \).
Section 2.6  Related Rates
Tips and Tools for Problem Solving

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Although we made changes to the section exercises, no changes were made based on the data.

Capstone
Page 156, Exercise 38  You can use this exercise to review the following concepts.

- Finding a related rate
- Using graphical reasoning

You can use this exercise to use a graphical approach to determining the sign of a related rate. Because you do not have to find any derivatives, you can use the graphs to reinforce the concept of related rates.

The graphs in this exercise are available as transparencies. Use each graph of $f$ to show that when the rate of change of $x$ is negative ($x$ is decreasing) that the rate of change of $y$ is positive ($y$ is increasing) in graph (i) and negative ($y$ is decreasing) in graph (ii). Then use the graphs to show that when the rate of change of $y$ is positive ($y$ is increasing) that the rate of change of $x$ is negative ($x$ is decreasing) in graph (i) and positive ($x$ is increasing) in graph (ii).

Solution

(i)  (a)  \( \frac{dx}{dt} \) negative \Rightarrow \( \frac{dy}{dt} \) positive

(b)  \( \frac{dy}{dt} \) positive \Rightarrow \( \frac{dx}{dt} \) negative

(ii) (a)  \( \frac{dx}{dt} \) negative \Rightarrow \( \frac{dy}{dt} \) negative

(b)  \( \frac{dy}{dt} \) positive \Rightarrow \( \frac{dx}{dt} \) positive

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Chapter 2 Project

Timing a Handoff

You are a competitive bicyclist. During a race, you bike at a constant velocity of $k$ meters per second. A chase car waits for you at the ten-mile mark of a course. When you cross the ten-mile mark, the car immediately accelerates to catch you. The position function of the chase car is given by the equation $s(t) = \frac{15}{4} t^2 - \frac{5}{12} t^3$, for $0 \leq t \leq 6$, where $t$ is the time in seconds and $s$ is the distance traveled in meters. When the car catches you, you and the car are traveling at the same velocity, and the driver hands you a cup of water while you continue to bike at $k$ meters per second.

Exercises

1. Write an equation that represents your position $s$ (in meters) at time $t$ (in seconds).

2. Use your answer to Exercise 1 and the given information to write an equation that represents the velocity $k$ at which the chase car catches you in terms of $t$.

3. Find the velocity function of the car.

4. Use your answers to Exercises 2 and 3 to find how many seconds it takes the chase car to catch you.

5. What is your velocity when the car catches you?

6. Use a graphing utility to graph the chase car’s position function and your position function in the same viewing window.

7. Find the point of intersection of the two graphs in Exercise 6. What does this point represent in the context of the problem?

8. Describe the graphs in Exercise 6 at the point of intersection. Why is this important for a successful handoff?

9. Suppose you bike at a constant velocity of 9 meters per second and the chase car’s position function is unchanged.
   (a) Use a graphing utility to graph the chase car’s position function and your position function in the same viewing window.
   (b) In this scenario, how many times will the chase car be in the same position as you after the 10-mile mark?
   (c) In this scenario, would the driver of the car be able to successfully handoff a cup of water to you? Explain.

10. Suppose you bike at a constant velocity of 8 meters per second and the chase car’s position function is unchanged.
    (a) Use a graphing utility to graph the chase car’s position function and your position function in the same viewing window.
    (b) In this scenario, how many times will the chase car be in the same position as you after the 10-mile mark?
    (c) In this scenario, why might it be difficult for the driver of the chase car to successfully handoff a cup of water to you? Explain.