INSTRUCTOR'S MANUAL

APPLIED PARTIAL DIFFERENTIAL EQUATIONS
with Fourier Series and Boundary Value Problems

Fourth Edition

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Preface

In this manual there are solutions to most of the starred exercises of APPLIED PARTIAL DIFFERENTIAL EQUATIONS with Fourier Series and Boundary Value Problems, Fourth Edition, by Richard Haberman.

Over 1000 exercises of varying difficulty form an essential part of this text. It is hoped that these approximately 250 selected solutions will be useful for instructors and those contemplating adopting this text.

I would like to express my appreciation to Shari Webster and Nita Blanscet for the preparation of this manual using LaTeX.

Richard Haberman

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Chapter 1. Heat Equation

Section 1.2

1.2.9 (d) Circular cross section means that $P = 2\pi r, A = \pi r^2$, and thus $P/A = 2/r$, where r is the radius. Also $\gamma = 0$.

1.2.9 (e) $u(x,t) = u(t)$ implies that

$$c\rho\, \frac{du}{dt} = -\frac{2h}{r}u\,.$$

The solution of this first-order linear differential equation with constant coefficients, which satisfies the initial condition $u(0) = u_0$, is

$$u(t) = u_0 \exp\left[-\frac{2h}{c\rho r}t\right].$$

Section 1.3

1.3.2 $\partial u/\partial x$ is continuous if $K_0(x_0-) = K_0(x_0+)$, that is, if the conductivity is continuous.

Section 1.4

1.4.1 (a) Equilibrium satisfies (1.4.14), $d^2u/dx^2 = 0$, whose general solution is (1.4.17), $u = c_1 + c_2 x$. The boundary condition $u(0) = 0$ implies $c_1 = 0$ and $u(L) = T$ implies $c_2 = T/L$ so that $u = Tx/L$.

1.4.1 (d) Equilibrium satisfies (1.4.14), $d^2u/dx^2 = 0$, whose general solution (1.4.17), $u = c_1 + c_2 x$. From the boundary conditions, $u(0) = T$ yields $T = c_1$ and $du/dx(L) = \alpha$ yields $\alpha = c_2$. Thus $u = T + \alpha x$.

1.4.1 (f) In equilibrium, (1.2.9) becomes $d^2u/dx^2 = -Q/K_0 = -x^2$, whose general solution (by integrating twice) is $u = -x^4/12 + c_1 + c_2 x$. The boundary condition $u(0) = T$ yields $c_1 = T$, while $du/dx(L) = 0$ yields $c_2 = L^3/3$. Thus $u = -x^4/12 + L^3x/3 + T$.

1.4.1 (h) Equilibrium satisfies $d^2u/dx^2 = 0$. One integration yields $du/dx = c_2$, the second integration yields the general solution $u = c_1 + c_2 x$.

$$\begin{aligned} x = 0:&\quad c_2 - (c_1 - T) = 0 \\ x = L:&\quad c_2 = \alpha \text{ and thus } c_1 = T + \alpha. \end{aligned}$$

Therefore, $u = (T + \alpha) + \alpha x = T + \alpha(x + 1)$.

1.4.7 (a) For equilibrium:

$$\frac{d^2u}{dx^2} = -1 \text{ implies } u = -\frac{x^2}{2} + c_1 x + c_2 \text{ and } \frac{du}{dx} = -x + c_1.$$

From the boundary conditions $\frac{du}{dx}(0) = 1$ and $\frac{du}{dx}(L) = \beta, c_1 = 1$ and $-L + c_1 = \beta$ which is consistent only if $\beta + L = 1$. If $\beta = 1 - L$, there is an equilibrium solution ($u = -\frac{x^2}{2} + x + c_2$). If $\beta \neq 1 - L$, there isn't an equilibrium solution. The difficulty is caused by the heat flow being specified at both ends and a source specified inside. An equilibrium will exist only if these three are in balance. This balance can be mathematically verified from conservation of energy:

$$\frac{d}{dt}\int_0^L c\rho u\, dx = -\frac{du}{dx}(0) + \frac{du}{dx}(L) + \int_0^L Q_0\, dx = -1 + \beta + L.$$

If $\beta + L = 1$, then the total thermal energy is constant and the initial energy = the final energy:

$$\int_0^L f(x)\, dx = \int_0^L \left(-\frac{x^2}{2} + x + c_2\right)\, dx, \quad \text{which determines } \quad c_2.$$

If $\beta + L \neq 1$, then the total thermal energy is always changing in time and an equilibrium is never reached.

Section 1.5

1.5.9 (a) In equilibrium, (1.5.14) using (1.5.19) becomes $\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$. Integrating once yields $r du/dr = c_1$ and integrating a second time (after dividing by r) yields $u = c_1 \ln r + c_2$. An alternate general solution is $u = c_1 \ln(r/r_1) + c_3$. The boundary condition $u(r_1) = T_1$ yields $c_3 = T_1$, while $u(r_2) = T_2$ yields $c_1 = (T_2 - T_1)/\ln(r_2/r_1)$. Thus, $u = \frac{1}{\ln(r_2/r_1)}[(T_2 - T_1)\ln r/r_1 + T_1 \ln(r_2/r_1)]$.

1.5.11 For equilibrium, the radial flow at $r = a$, $2\pi a\beta$, must equal the radial flow at $r = b$, $2\pi b$. Thus $\beta = b/a$.

1.5.13 From exercise 1.5.12, in equilibrium $\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$. Integrating once yields $r^2 du/dr = c_1$ and integrating a second time (after dividing by r^2) yields $u = -c_1/r + c_2$. The boundary conditions $u(4) = 80$ and $u(1) = 0$ yields $80 = -c_1/4 + c_2$ and $0 = -c_1 + c_2$. Thus $c_1 = c_2 = 320/3$ or $u = \frac{320}{3}\left(1 - \frac{1}{r}\right)$.

Chapter 2. Method of Separation of Variables

Section 2.3

2.3.1 (a) $u(r,t) = \phi(r)h(t)$ yields $\phi\frac{dh}{dt} = \frac{kh}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)$. Dividing by $k\phi h$ yields $\frac{1}{kh}\frac{dh}{dt} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda$ or $\frac{dh}{dt} = -\lambda kh$ and $\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda\phi$.

2.3.1 (c) $u(x,y) = \phi(x)h(y)$ yields $h\frac{d^2\phi}{dx^2} + \phi\frac{d^2h}{dy^2} = 0$. Dividing by ϕh yields $\frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\frac{1}{h}\frac{d^2h}{dy^2} = -\lambda$ or $\frac{d^2\phi}{dx^2} = -\lambda\phi$ and $\frac{d^2h}{dy^2} = \lambda h$.

2.3.1 (e) $u(x,t) = \phi(x)h(t)$ yields $\phi(x)\frac{dh}{dt} = kh(t)\frac{d^4\phi}{dx^4}$. Dividing by $k\phi h$, yields $\frac{1}{kh}\frac{dh}{dt} = \frac{1}{\phi}\frac{d^4\phi}{dx^4} = \lambda$.

2.3.1 (f) $u(x,t) = \phi(x)h(t)$ yields $\phi(x)\frac{d^2h}{dt^2} = c^2h(t)\frac{d^2\phi}{dx^2}$. Dividing by $c^2\phi h$, yields $\frac{1}{c^2h}\frac{d^2h}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$.

2.3.2 (b) $\lambda = (n\pi/L)^2$ with $L = 1$ so that $\lambda = n^2\pi^2,\ n = 1, 2, \ldots$

2.3.2 (d)

(i) If $\lambda > 0, \phi = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$. $\phi(0) = 0$ implies $c_1 = 0$, while $\frac{d\phi}{dx}(L) = 0$ implies $c_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$. Thus $\sqrt{\lambda}L = -\pi/2 + n\pi(n = 1, 2, \ldots)$.

(ii) If $\lambda = 0, \phi = c_1 + c_2x$. $\phi(0) = 0$ implies $c_1 = 0$ and $d\phi/dx(L) = 0$ implies $c_2 = 0$. Therefore $\lambda = 0$ is not an eigenvalue.

(iii) If $\lambda < 0$, let $\lambda = -s$ and $\phi = c_1\cosh\sqrt{s}x + c_2\sinh\sqrt{s}x$. $\phi(0) = 0$ implies $c_1 = 0$ and $d\phi/dx(L) = 0$ implies $c_2\sqrt{s}\cosh\sqrt{s}L = 0$. Thus $c_2 = 0$ and hence there are no eigenvalues with $\lambda < 0$.

2.3.2 (f) The simpliest method is to let $x' = x - a$. Then $d^2\phi/dx'^2 + \lambda\phi = 0$ with $\phi(0) = 0$ and $\phi(b-a) = 0$. Thus (from p. 46) $L = b - a$ and $\lambda = [n\pi/(b-a)]^2,\ n = 1, 2, \ldots$.

2.3.3 From (2.3.30), $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2t}$. The initial condition yields $2\cos\frac{3\pi x}{L} = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}$. From (2.3.35), $B_n = \frac{2}{L}\int_0^L 2\cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}\,dx$.

2.3.4 (a) Total heat energy $= \int_0^L c\rho uA\,dx = c\rho A\sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\frac{1-\cos n\pi}{\frac{n\pi}{L}}$, using (2.3.30) where B_n satisfies (2.3.35).

2.3.4 (b)

heat flux to right $= -K_0\partial u/\partial x$
total heat flow to right $= -K_0A\partial u/\partial x$
heat flow out at $x = 0 = K_0A\frac{\partial u}{\partial x}\Big|_{x=0}$
heat flow out $(x = L) = -K_0A\frac{\partial u}{\partial x}\Big|_{x=L}$

2.3.4 (c) From conservation of thermal energy, $\frac{d}{dt}\int_0^L u\,dx = k\frac{\partial u}{\partial x}\Big|_0^L = k\frac{\partial u}{\partial x}(L) - k\frac{\partial u}{\partial x}(0)$. Integrating from $t = 0$ yields

$$\underbrace{\int_0^L u(x,t)\,dx}_{\text{heat energy at } t} - \underbrace{\int_0^L u(x,0)\,dx}_{\text{initial heat energy}} = k\int_0^t \Bigg[\underbrace{\frac{\partial u}{\partial x}(L)}_{\text{integral of flow in at } x=L} - \underbrace{\frac{\partial u}{\partial x}(0)}_{\text{integral of flow out at } x=L}\Bigg]\,dx\quad.$$

2.3.8 (a) The general solution of $k\frac{d^2u}{dx^2} = \alpha u\ (\alpha > 0)$ is $u(x) = a\cosh\sqrt{\frac{\alpha}{k}}x + b\sinh\sqrt{\frac{\alpha}{k}}x$. The boundary condition $u(0) = 0$ yields $a = 0$, while $u(L) = 0$ yields $b = 0$. Thus $u = 0$.

2.3.8 (b) Separation of variables, $u = \phi(x)h(t)$ or $\phi\frac{dh}{dt} + \alpha\phi h = kh\frac{d^2\phi}{dx^2}$, yields two ordinary differential equations (divide by $k\phi h$): $\frac{1}{kh}\frac{dh}{dt} + \frac{\alpha}{k} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$. Applying the boundary conditions, yields the eigenvalues $\lambda = (n\pi/L)^2$ and corresponding eigenfunctions $\phi = \sin\frac{n\pi x}{L}$. The time-dependent part are exponentials, $h = e^{-\lambda kt}e^{-\alpha t}$. Thus by superposition, $u(x,t) = e^{-\alpha t}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}$, where the initial conditions $u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ yields $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$. As $t \to \infty$, $u \to 0$, the only equilibrium solution.

2.3.9 (a) If $\alpha < 0$, the general equilibrium solution is $u(x) = a\cos\sqrt{\frac{-\alpha}{k}}x + b\sin\sqrt{\frac{-\alpha}{k}}x$. The boundary condition $u(0) = 0$ yields $a = 0$, while $u(L) = 0$ yields $b\sin\sqrt{\frac{-\alpha}{k}}L = 0$. Thus if $\sqrt{\frac{-\alpha}{k}}L \neq n\pi, u = 0$ is the only equilibrium solution. However, if $\sqrt{\frac{-\alpha}{k}}L = n\pi$, then $u = A\sin\frac{n\pi x}{L}$ is an equilibrium solution.

2.3.9 (b) Solution obtained in 2.3.8 is correct. If $-\frac{\alpha}{k} = \left(\frac{\pi}{L}\right)^2, u(x,t) \to b_1\sin\frac{\pi x}{L}$, the equilibrium solution. If $-\frac{\alpha}{k} < \left(\frac{\pi}{L}\right)^2$, then $u \to 0$ as $t \to \infty$. However, if $-\frac{\alpha}{k} > \left(\frac{\pi}{L}\right)^2, u \to \infty$ (if $b_1 \neq 0$). Note that $b_1 > 0$ if $f(x) \geq 0$. Other more unusual events can occur if $b_1 = 0$. [Essentially, the other possible equilibrium solutions are unstable.]

Section 2.4

2.4.1 The solution is given by (2.4.19), where the coefficients satisfy (2.4.21) and hence (2.4.23-24).

(a) $A_0 = \frac{1}{L}\int_{L/2}^{L} 1dx = \frac{1}{2}, A_n = \frac{2}{L}\int_{L/2}^{L}\cos\frac{n\pi x}{L}dx = \frac{2}{L}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{L/2}^{L} = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$

(b) by inspection $A_0 = 6, A_3 = 4$, others $= 0$.

(c) $A_0 = \frac{-2}{L}\int_0^L \sin\frac{\pi x}{L}\,dx = \frac{2}{\pi}\cos\frac{\pi x}{L}\Big|_0^L = \frac{2}{\pi}(1 - \cos\pi) = 4/\pi, A_n = \frac{-4}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx$

(d) by inspection $A_8 = -3$, others $= 0$.

2.4.3 Let $x' = x - \pi$. Then the boundary value problem becomes $d^2\phi/dx'^2 = -\lambda\phi$ subject to $\phi(-\pi) = \phi(\pi)$ and $d\phi/dx'(-\pi) = d\phi/dx'(\pi)$. Thus, the eigenvalues are $\lambda = (n\pi/L)^2 = n^2\pi^2$, since $L = \pi, n = 0, 1, 2, \ldots$ with the corresponding eigenfunctions being both $\sin n\pi x'/L = \sin n(x-\pi) = (-1)^n \sin nx => \sin nx$ and $\cos n\pi x'/L = \cos n(x - \pi) = (-1)^n \cos nx => \cos nx$.

Section 2.5

2.5.1 (a) Separation of variables, $u(x,y) = h(x)\phi(y)$, implies that $\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = -\lambda$. Thus $d^2h/dx^2 = -\lambda h$ subject to $h'(0) = 0$ and $h'(L) = 0$. Thus as before, $\lambda = (n\pi/L)^2, n = 0, l, 2, \ldots$ with $h(x) = \cos n\pi x/L$. Furthermore, $\frac{d^2\phi}{dy^2} = \lambda\phi = \left(\frac{n\pi}{L}\right)^2\phi$ so that

$n = 0 : \phi = c_1 + c_2 y$, where $\phi(0) = 0$ yields $c_1 = 0$

$n \neq 0 : \phi = c_1\cosh\frac{n\pi y}{L} + c_2\sinh\frac{n\pi y}{L}$, where $\phi(0) = 0$ yields $c_1 = 0$.

The result of superposition is

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}.$$

The nonhomogeneous boundary condition yields

$$f(x) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi H}{L}\cos\frac{n\pi x}{L},$$

so that

$$A_0 H = \frac{1}{L}\int_0^L f(x)\,dx \text{ and } A_n\sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

2.5.1 (c) Separation of variables, $u = h(x)\phi(y)$, yields $\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = \lambda$. The boundary conditions $\phi(0) = 0$ and $\phi(H) = 0$ yield an eigenvalue problem in y, whose solution is $\lambda = (n\pi/H)^2$ with $\phi = \sin n\pi y/H, n = 1, 2, 3, \ldots$ The solution of the x-dependent equation is $h(x) = \cosh n\pi x/H$ using $dh/dx(0) = 0$. By superposition:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi x}{H} \sin \frac{n\pi y}{H}.$$

The nonhomogeneous boundary condition at $x = L$ yields $g(y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi L}{H} \sin \frac{n\pi y}{H}$, so that A_n is determined by $A_n \cosh \frac{n\pi L}{H} = \frac{2}{H}\int_0^H g(y) \sin \frac{n\pi y}{H}\, dy$.

2.5.1 (e) Separation of variables, $u = \phi(x)h(y)$, yields the eigenvalues $\lambda = (n\pi/L)^2$ and corresponding eigenfunctions $\phi = \sin n\pi x/L, n = 1, 2, 3, \ldots$ The y-dependent differential equation, $\frac{d^2h}{dy^2} = \left(\frac{n\pi}{L}\right)^2 h$, satisfies $h(0) - \frac{dh}{dy}(0) = 0$. The general solution $h = c_1 \cosh \frac{n\pi y}{L} + c_2 \sinh \frac{n\pi y}{L}$ obeys $h(0) = c_1$, while $\frac{dh}{dy} = \frac{n\pi}{L}\left(c_1 \sinh \frac{n\pi y}{L} + c_2 \cosh \frac{n\pi y}{L}\right)$ obeys $\frac{dh}{dy}(0) = c_2 \frac{n\pi}{L}$. Thus, $c_1 = c_2 \frac{n\pi}{L}$ and hence $h_n(y) = \cosh \frac{n\pi y}{L} + \frac{L}{n\pi} \sinh \frac{n\pi y}{L}$. Superposition yields

$$u(x,y) = \sum_{n=1}^{\infty} A_n h_n(y) \sin n\pi x/L,$$

where A_n is determined from the boundary condition, $f(x) = \sum_{n=1}^{\infty} A_n h_n(H) \sin n\pi x/L$, and hence

$$A_n h_n(H) = \frac{2}{L}\int_0^L f(x) \sin n\pi x/L \, dx .$$

2.5.2 (a) From physical reasoning (or exercise 1.5.8), the total heat flow across the boundary must equal zero in equilibrium (without sources, i.e. Laplace's equation). Thus $\int_0^L f(x)\, dx = 0$ for a solution.

2.5.3 In order for u to be bounded as $r \to \infty, c_1 = 0$ in (2.5.43) and $\bar{c}_2 = 0$ in (2.5.44). Thus,

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{-n} \cos n\theta + \sum_{n=1}^{\infty} B_n r^{-n} \sin n\theta.$$

(a) The boundary condition yields $A_0 = \ln 2, A_3 a^{-3} = 4$, other $A_n = 0, B_n = 0$.

(b) The boundary conditions yield (2.5.46) with a^{-n} replacing a^n. Thus, the coefficients are determined by (2.5.47) with a^n replaced by a^{-n}

2.5.4 By substituting (2.5.47) into (2.5.45) and interchanging the orders of summation and integration

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta}) \left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n (\cos n\theta \cos n\bar{\theta} + \sin n\theta \sin n\bar{\theta})\right] d\bar{\theta}.$$

Noting the trigonometric addition formula and $\cos z = R_e[e^{iz}]$, we obtain

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta}) \left[-\frac{1}{2} + Re\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\bar{\theta})}\right] d\bar{\theta}.$$

Summing the geometric series enables the bracketed term to be replaced by

$$-\frac{1}{2} + Re\frac{1}{1 - \frac{r}{a}e^{i(\theta-\bar{\theta})}} = -\frac{1}{2} + \frac{1 - \frac{r}{a}\cos(\theta-\bar{\theta})}{1 + \frac{r^2}{a^2} - \frac{2r}{a}\cos(\theta-\bar{\theta})} = \frac{\frac{1}{2} - \frac{1}{2}\frac{r^2}{a^2}}{1 + \frac{r^2}{a^2} - \frac{2r}{a}\cos(\theta-\bar{\theta})}.$$

2.5.5 (a) The eigenvalue problem is $d^2\phi/d\theta^2 = -\lambda\phi$ subject to $d\phi/d\theta(0) = 0$ and $\phi(\pi/2) = 0$. It can be shown that $\lambda > 0$ so that $\phi = \cos\sqrt{\lambda}\theta$ where $\phi(\pi/2) = 0$ implies that $\cos\sqrt{\lambda}\pi/2 = 0$ or $\sqrt{\lambda}\pi/2 = -\pi/2 + n\pi, n = 1, 2, 3, \ldots$ The eigenvalues are $\lambda = (2n-1)^2$. The radially dependent term satisfies (2.5.40), and hence the boundedness condition at $r = 0$ yields $G(r) = r^{2n-1}$. Superposition yields

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n-1} \cos(2n-1)\theta.$$

The nonhomogeneous boundary condition becomes

$$f(\theta) = \sum_{n=1}^{\infty} A_n \cos(2n-1)\theta \quad \text{or} \quad A_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\cos(2n-1)\theta\, d\theta.$$

2.5.5 (c) The boundary conditions of (2.5.37) must be replaced by $\phi(0) = 0$ and $\phi(\pi/2) = 0$. Thus, $L = \pi/2$, so that $\lambda = (n\pi/L)^2 = (2n)^2$ and $\phi = \sin\frac{n\pi\theta}{L} = \sin 2n\theta$. The radial part that remains bounded at $r = 0$ is $G = r^{\sqrt{\lambda}} = r^{2n}$. By superposition,

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin 2n\theta\ .$$

To apply the nonhomogeneous boundary condition, we differentiate with respect to r:

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} A_n (2n) r^{2n-1} \sin 2n\theta\ .$$

The bc at $r = 1$, $f(\theta) = \sum_{n=1}^{\infty} 2nA_n \sin 2n\theta$, determines $A_n, 2nA_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin 2n\theta\; d\theta$.

2.5.6 (a) The boundary conditions of (2.5.37) must be replaced by $\phi(0) = 0$ and $\phi(\pi) = 0$. Thus $L = \pi$, so that the eigenvalues are $\lambda = (n\pi/L)^2 = n^2$ and corresponding eigenfunctions $\phi = \sin n\pi\theta/L = \sin n\theta, n = 1, 2, 3, \ldots$ The radial part which is bounded at $r = 0$ is $G = r^{\sqrt{\lambda}} = r^n$. Thus by superposition

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta\ .$$

The bc at $r = a$, $g(\theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta$, determines $A_n, A_n a^n = \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin n\theta\; d\theta$.

2.5.7 (b) The boundary conditions of (2.5.37) must be replaced by $\phi'(0) = 0$ and $\phi'(\pi/3) = 0$. This will yield a cosine series with $L = \pi/3, \lambda = (n\pi/L)^2 = (3n)^2$ and $\phi = \cos n\pi\theta/L = \cos 3n\theta, n = 0, 1, 2, \ldots$. The radial part which is bounded at $r = 0$ is $G = r^{\sqrt{\lambda}} = r^{3n}$. Thus by superposition

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{3n} \cos 3n\theta\ .$$

The boundary condition at $r = a$, $g(\theta) = \sum_{n=0}^{\infty} A_n a^{3n}\cos 3n\theta$, determines A_n: $A_0 = \frac{3}{\pi}\int_0^{\pi/3} g(\theta)\; d\theta$ and $(n \neq 0) A_n a^{3n} = \frac{6}{\pi}\int_0^{\pi/3} g(\theta)\cos 3n\theta\; d\theta$.

2.5.8 (a) There is a full Fourier series in θ. It is easier (but equivalent) to choose radial solutions that satisfy the corresponding homogeneous boundary condition. Instead of r^n and r^{-n} (1 and $\ln r$ for $n = 0$), we choose $\phi_1(r)$ such that $\phi_1(a) = 0$ and $\phi_2(r)$ such that $\phi_2(b) = 0$:

$$\phi_1(r) = \begin{cases} \ln(r/a) & n = 0 \\ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n & n \neq 0 \end{cases} \quad \phi_2(r) = \begin{cases} \ln(r/b) & n = 0 \\ \left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n & n \neq 0 \end{cases}.$$

Then by superposition

$$u(r,\theta)=\sum_{n=0}^{\infty}\cos n\theta\,[A_n\phi_1(r)+B_n\phi_2(r)]+\sum_{n=1}^{\infty}\sin n\theta\,[C_n\phi_1(r)+D_n\phi_2(r)].$$

The boundary conditions at $r=a$ and $r=b$,

$$f(\theta)=\sum_{n=0}^{\infty}\cos n\theta\,[A_n\phi_1(a)+B_n\phi_2(a)]+\sum_{n=1}^{\infty}\sin n\theta\,[C_n\phi_1(a)+D_n\phi_2(a)]$$

$$g(\theta)=\sum_{n=0}^{\infty}\cos n\theta\,[A_n\phi_1(b)+B_n\phi_2(b)]+\sum_{n=1}^{\infty}\sin n\theta\,[C_n\phi_1(b)+D_n\phi_2(b)]$$

easily determine A_n, B_n, C_n, D_n since $\phi_1(a)=0$ and $\phi_2(b)=0: D_n\phi_2(a)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\sin n\theta\,d\theta$, etc.

2.5.9 (a) The boundary conditions of (2.5.37) must be replaced by $\phi(0)=0$ and $\phi(\pi/2)=0$. This is a sine series with $L=\pi/2$ so that $\lambda=(n\pi/L)^2=(2n)^2$ and the eigenfunctions are $\phi=\sin n\pi\theta/L=\sin 2n\theta, n=1,2,3,\ldots$. The radial part which is zero at $r=a$ is $G=(r/a)^{2n}-(a/r)^{2n}$. Thus by superposition,

$$u(r,\theta)=\sum_{n=1}^{\infty}A_n\left[\left(\frac{r}{a}\right)^{2n}-\left(\frac{a}{r}\right)^{2n}\right]\sin 2n\theta.$$

The nonhomogeneous boundary condition, $f(\theta)=\sum_{n=1}^{\infty}A_n\left[\left(\frac{b}{a}\right)^{2n}-\left(\frac{a}{b}\right)^{2n}\right]\sin 2n\theta$, determines A_n : $A_n\left[\left(\frac{b}{a}\right)^{2n}-\left(\frac{a}{b}\right)^{2n}\right]=\frac{4}{\pi}\int_0^{\pi/2}f(\theta)\sin 2n\theta\,d\theta$.

2.5.9 (b) The two homogeneous boundary conditions are in r, and hence $\phi(r)$ must be an eigenvalue problem. By separation of variables, $u=\phi(r)G(\theta), d^2G/d\theta^2=\lambda G$ and $r^2\frac{d^2\phi}{dr^2}+r\frac{d\phi}{dr}+\lambda\phi=0$. The radial equation is equidimensional (see p.78) and solutions are in the form $\phi=r^p$. Thus $p^2=-\lambda$ (with $\lambda>0$) so that $p=\pm i\sqrt{\lambda}$. $r^{\pm i\sqrt{\lambda}}=e^{\pm i\sqrt{\lambda}\ln r}$. Thus real solutions are $\cos(\sqrt{\lambda}\ln r)$ and $\sin(\sqrt{\lambda}\ln r)$. It is more convenient to use independent solutions which simplify at $r=a, \cos[\sqrt{\lambda}\ln(r/a)]$ and $\sin[\sqrt{\lambda}\ln(r/a)]$. Thus the general solution is

$$\phi=c_1\cos[\sqrt{\lambda}\ln(r/a)]+c_2\sin[\sqrt{\lambda}\ln(r/a)].$$

The homogeneous condition $\phi(a)=0$ yields $0=c_1$, while $\phi(b)=0$ implies $\sin[\sqrt{\lambda}\ln(r/a)]=0$. Thus $\sqrt{\lambda}\ln(b/a)=n\pi,\ n=1,2,3,\ldots$ and the corresponding eigenfunctions are $\phi=\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]$. The solution of the θ -equation satisfying $G(0)=0$ is $G=\sinh\sqrt{\lambda}\theta=\sinh\frac{n\pi\theta}{\ln(b/a)}$. Thus by superposition

$$u=\sum_{n=1}^{\infty}A_n\sinh\frac{n\pi\theta}{\ln(b/a)}\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right].$$

The nonhomogeneous boundary condition,

$$f(r)=\sum_{n=1}^{\infty}A_n\sinh\frac{n\pi^2}{2\ln(b/a)}\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right],$$

will determine A_n. One method (for another, see exercise 5.3.9) is to let $z=\ln(r/a)/\ln(b/a)$. Then $a<r<b$, lets $0<z<1$. This is a sine series in z (with $L=1$) and hence

$$A_n\sinh\frac{n\pi^2}{2\ln(b/a)}=2\int_0^1 f(r)\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]dz.$$

But $dz=dr/r\ln(b/a)$. Thus

$$A_n\sinh\frac{n\pi^2}{2\ln(b/a)}=\frac{2}{\ln(b/a)}\int_0^1 f(r)\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]dr/r.$$

Chapter 3. Fourier Series

Section 3.2

3.2.2 (a) $x \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x/L + \sum_{n=1}^{\infty} b_n \sin n\pi x/L$. From (3.2.2), $a_0 = 0$ since $f(x)$ is odd, $(n \neq 0) a_n = 0$ since $f(x)$ is odd, and $b_n = \frac{1}{L}\int_{-L}^{L} x \sin n\pi x/L \, dx = \frac{2L}{n\pi}(-1)^{n+1}$.

3.2.2 (c) $\sin \pi x/L \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x/L + \sum_{n=1}^{\infty} b_n \sin n\pi x/L$. By inspection, $b_1 = 1$, all others $= 0$.

3.2.2 (f) From (3.2.2),

$$\begin{aligned} a_0 &= \tfrac{1}{2L}\textstyle\int_0^L dx = 1/2 \\ (n \neq 0) \quad a_n &= \tfrac{1}{L}\textstyle\int_0^L \cos n\pi x/L \, dx = 0 \\ b_n &= \tfrac{1}{L}\textstyle\int_0^L \sin n\pi x/L \, dx = \tfrac{-1}{n\pi}\cos \tfrac{n\pi x}{L}\Big|_0^L = \tfrac{1-\cos n\pi}{n\pi} \end{aligned}$$

Thus $b_n = 2/n\pi$, n odd, but $b_n = 0$, n even.

3.3.2 (d) From (3.3.6), $B_n = \frac{2}{L}\int_0^{L/2} \sin n\pi x/L \, dx = \frac{-2}{n\pi}\cos n\pi x/L\Big|_0^{L/2} = \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$.

3.3.10 $f(-x) = \begin{cases} x^2 & -x < 0 \quad (\text{or } x > 0) \\ e^x & -x > 0 \quad (\text{or } x < 0). \end{cases}$ Thus.

$$f_e(x) = \tfrac{1}{2}[f(x) + f(-x)] = \tfrac{1}{2}\begin{cases} x^2 + e^x & x < 0 \\ x^2 + e^{-x} & x > 0 \end{cases}$$
$$f_0(x) = \tfrac{1}{2}[f(x) - f(-x)] = \tfrac{1}{2}\begin{cases} x^2 - e^x & x < 0 \\ e^{-x} - x^2 & x > 0 \end{cases}$$

3.3.13 $b_n = \frac{2}{L}\int_0^L f(x)\sin n\pi x/L \, dx$, given that $f(x)$ is even around $L/2$. Note (perhaps by graphing) that $\sin n\pi x/L$ is odd around $L/2$ for n even. Thus $f(x)\sin n\pi x/L$ is odd around $L/2$ for n even, and hence $b_n = 0$ for n even.

Section 3.4

3.4.1 (a) $\int_a^b = \int_a^{c-} + \int_{c+}^b$. Thus

$$\int_a^b u\frac{dv}{dx}\,dx = uv\Big|_a^{c-} + uv\Big|_{c+}^b - \int_a^b v\frac{du}{dx}\,dx = uv\Big|_a^b + uv\Big|_{c+}^{c-} - \int_a^b v\frac{du}{dx}\,dx.$$

3.4.3 (a) We want to determine the sine coefficients of df/dx: $\frac{df}{dx} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$, where the cosine coefficients of f are given

$$f = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad (n \neq 0) \;\; a_n = \frac{2}{L}\int_0^L f\cos\frac{n\pi x}{L}\,dx.$$

Here by integration by parts

$$b_n = \frac{2}{L}\int_0^L \frac{df}{dx}\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\left[f\sin\frac{n\pi x}{L}\Big|_0^{x_0-} + f\sin\frac{n\pi x}{L}\Big|_{x_0+}^L - \frac{n\pi}{L}\int_0^L f\cos\frac{n\pi x}{L}\,dx\right].$$

Thus $b_n = \frac{2}{L}\sin\frac{n\pi x_0}{L}(\alpha - \beta) - \frac{n\pi}{L}a_n$.

3.4.9 $\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right) b_n \cos\frac{n\pi x}{L}$ since $u(0)=0$ and $u(L)=0$. $\frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n \sin\frac{n\pi x}{L}$. Thus from p. 119, $\sum_{n=1}^{\infty}\frac{db_n}{dt}\sin\frac{n\pi x}{L} \sim -k\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n\sin\frac{n\pi x}{L} + q$. Thus $\frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n = \frac{2}{L}\int_0^L q\sin\frac{n\pi x}{L}\,dx$.

3.4.12 The eigenfunctions of the related homogeneous problem are $\cos n\pi x/L, n = 0,1,2,\ldots$. Thus $u \sim \sum_{n=0}^{\infty} A_n(t)\cos n\pi x/L$, which can be differentiated (if u is continuous) since it is a cosine series: $\partial u/\partial x \sim \sum_{n=0}^{\infty} A_n(-n\pi/L)\sin\frac{n\pi x}{L}$. This can be differentiated again (if $\partial u/\partial x$ is continuous) only because $\partial u/\partial x(0) = 0$ and $\partial u/\partial x(L) = 0$: $\partial^2 u/\partial x^2 \sim -\sum_{n=0}^{\infty} A_n(n\pi/L)^2\cos n\pi x/L$. Thus from p. 119

$$\sum_{n=0}^{\infty}\left[\frac{dA_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 A_n\right]\cos\frac{n\pi x}{L} = e^{-t} + e^{-2t}\cos\frac{3\pi x}{L}.$$

The right hand side is a simple cosine series (with only two non-zero terms). Thus

$$\frac{dA_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 A_n = \begin{cases} e^{-t} & n=0 \\ e^{-2t} & n=3 \\ 0 & \text{otherwise}. \end{cases}$$

The initial conditions are $A_0(0) = \frac{1}{L}\int_0^L f(x)\,dx$ and $(n \neq 0)$ $A_n(0) = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$. The solution of the differential equations are

$$\begin{aligned} n \neq 0,3 \quad A_n(t) &= A_n(0)e^{-k(n\pi/L)^2 t} \\ A_0(t) &= A_0(0) + 1 - e^{-t}, \text{ obtained by integration} \\ A_3(t) &= A_3(0)e^{-k(n\pi/L)^2 t} + \frac{e^{-2t} - e^{-k(n\pi/L)^2 t}}{k\left(\frac{3\pi}{L}\right)^2 - 2}, \end{aligned}$$

obtained by using the method of undetermined coefficients (judicious guessing) for the particular solution. This works if e^{-2t} is not a homogeneous solution, i.e., $-2 \neq -k(3\pi/L)^2$.

Section 3.5

3.5.4 (a) Using 3.4.13 with $f(x) = \cosh x (f(0) = 1, f(L) = \cosh L)$,
$\sinh x \sim \frac{1}{L}(\cosh L - 1) + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}b_n + \frac{2}{L}((-1)^n\cosh L - 1)\right]\cos\frac{n\pi x}{L}$. Since this is a cosine series, it may be differentiated

$$\cosh x \sim -\sum_{n=1}^{\infty}\left[\left(\frac{n\pi}{L}\right)^2 b_n + \frac{2n\pi}{L^2}((-1)^n\cosh L - 1)\right]\sin\frac{n\pi x}{L}.$$

Thus $b_n = -\left(\frac{n\pi}{L}\right)^2 b_n - \frac{2n\pi}{L^2}[(-1)^n\cosh L - 1]$ or $b_n = \frac{2n\pi}{L^2}\frac{1-(-1)^n\cosh L}{1+(n\pi/L)^2}$.

3.5.4 (b) Integrating yields $\sinh x = A_0 + \sum_{n=1}^{\infty} -\frac{L}{n\pi}b_n\cos n\pi x/L$, where $A_0 = \frac{1}{L}\int_0^L \sinh x\,dx = \frac{1}{L}(\cosh L - 1)$. Integrating again yields $\cosh x - 1 = A_0 x + \sum_{n=1}^{\infty} -\left(\frac{L}{n\pi}\right)^2 b_n \sin n\pi x/L$. Thus

$$\sum_{n=1}^{\infty} b_n\left[1 + \left(\frac{L}{n\pi}\right)^2\right]\sin n\pi x/L = 1 + A_0 x.$$

Using (3.3.8) and (3.3.12) $b_n\left[1 + \left(\frac{L}{n\pi}\right)^2\right] = \frac{2}{n\pi}[1-(-1)^n] + \frac{1}{L}(\cosh L - 1)\frac{2L}{n\pi}(-1)^{n+1}$ or

$$b_n = \frac{2}{n\pi}\frac{1-(-1)^n\cosh L}{1+\left(\frac{L}{n\pi}\right)^2}.$$

3.5.7 Evaluate (3.5.6) at $x = L/2$:

$$\frac{L^2}{8} = \frac{L^2}{4} - \frac{4L^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right) \text{ or } 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} = \frac{L^2/8}{4L^2/\pi^3} = \pi^3/32.$$

Section 3.6

3.6.1 The complex Fourier coefficient is defined by (3.6.7):

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}dx = \frac{1}{2L\triangle}\int_{x_0}^{x_0+\triangle} e^{in\pi x/L}dx \,.$$

Thus

$$c_n = \frac{1}{2L\triangle}\frac{L}{in\pi}e^{in\pi x/L}\Big|_{x_0}^{x_0+\triangle} = \frac{1}{2in\pi\triangle}e^{in\pi x_0/L}(e^{in\pi\triangle/L}-1)\,.$$

Equivalently,

$$c_n = \frac{1}{2in\pi\triangle}e^{in\pi(x_0+\triangle/2)/L}(e^{in\pi\triangle/2L}-e^{-in\pi\triangle/2L}) \quad \text{or}$$

$$c_n = \frac{1}{n\pi\triangle}e^{in\pi(x_0+\triangle/2)/L}\sin(n\pi\triangle/2L)\,.$$

Chapter 4. Vibrating Strings and Membranes

Section 4.4

4.4.1 (a) Natural frequencies are $c\sqrt{\lambda}$, but $\lambda = (n\pi/L)^2$. Thus frequencies are $n\pi c/L, n = 1, 2, 3, ...$

4.4.1 (b) Natural frequencies are $c\sqrt{\lambda}$. The boundary condition $\phi(0) = 0$ implies $\phi = c_1 \sin\sqrt{\lambda}x$, while $d\phi/dx(H) = 0$ yields $\sqrt{\lambda}H = (m - \frac{1}{2})\pi$ with $m = 1, 2, 3$. Thus the frequencies are $(m - \frac{1}{2})\pi c/H$ and the eigenfunctions are $\sin(m - \frac{1}{2})\pi x/H$.

4.4.2 (c) By separation of variables, $u = \phi(x)h(t)$, $\frac{d^2h}{dt^2} = -\lambda h$ and $T_0\frac{d^2\phi}{dx^2} + (\alpha + \lambda\rho_0)\phi = 0$. With $\phi(0) = 0$ and $\phi(L) = 0, (\alpha + \lambda\rho_0)/T_0 = (n\pi/L)^2, n = 1, 2, 3, \ldots$ and $\phi = \sin n\pi x/L$. In general $h(t)$ involves a linear combination of $\sin\sqrt{\lambda}t$ and $\cos\sqrt{\lambda}t$, but the homogeneous initial condition $u(x, 0) = 0$ implies there are no cosines. Thus by superposition

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}t \sin n\pi x/L,$$

where the frequencies of vibration are $\sqrt{\lambda_n} = \sqrt{\frac{(n\pi/L)^2T_0 - \alpha}{\rho_0}}$. The other initial condition, $f(x) = \sum_{n=1}^{\infty} A_n\sqrt{\lambda_n}\sin n\pi x/L$, determines A_n

$$A_n\sqrt{\lambda_n} = \frac{2}{L}\int_0^L f(x)\sin n\pi x/L\, dx.$$

4.4.3 (b) By separation of variables, $u = \phi(x)h(t)$, $\frac{\rho_0 h'' + \beta h'}{hT_0} = \frac{\phi''}{\phi} = -\lambda$. The boundary conditions $\phi(0) = 0$ and $\phi(L) = 0$ yield $\lambda = (n\pi/L)^2$ with $\phi = \sin n\pi x/L$, $n = 1, 2, 3, \ldots$. The time-dependent equation has constant coefficients,

$$\rho_0 h'' + \beta h' + \left(\frac{n\pi}{L}\right)^2 T_0 h = 0,$$

and hence can be solved by substitution $h = e^{rt}$. This yields the quadratic equation

$$\rho_0 r^2 + \beta r + \left(\frac{n\pi}{L}\right)^2 T_0 = 0,$$

whose roots are

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 T_0(n\pi/L)^2}}{2\rho_0}.$$

Since $\beta^2 < 4\rho_0 T_0(\pi/L)^2$, the discriminant is < 0 for all n:

$$r = -\frac{\beta}{2\rho_0} + iw_n, \text{ where } w_n = \sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}.$$

Real solutions are $h = e^{-\beta t/2\rho_0}(\sin w_n t, \cos w_n t)$. Thus by superposition

$$u = e^{-\beta t/2\rho_0}\sum_{n=1}^{\infty}(a_n \cos w_n t + b_n \sin w_n t)\sin\frac{n\pi x}{L}.$$

The initial condition $u(x, 0) = f(x)$ determines a_n, $a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$, while $\frac{\partial u}{\partial t}(x, 0) = g(x)$ is a little more complicated, $g(x) = \sum_{n=1}^{\infty} b_n w_n \sin\frac{n\pi x}{L} - \frac{\beta}{2\rho_0}\underbrace{\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}}_{f(x)}$, and thus

$$b_n w_n = \frac{\beta a_n}{2\rho_0} + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\, dx.$$