CHAPTER 2
Limits and Continuity

EXERCISE SET 2.1

1. **(a)** 0 **(b)** 0 **(c)** 0 **(d)** 3

2. **(a)** $+\infty$ **(b)** $+\infty$ **(c)** $+\infty$ **(d)** undef

3. **(a)** $-\infty$ **(b)** $-\infty$ **(c)** $-\infty$ **(d)** 1

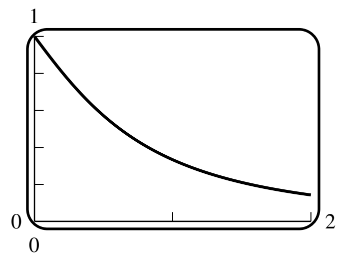
4. **(a)** 1 **(b)** $-\infty$ **(c)** does not exist **(d)** -2

5. for all $x_0 \neq -4$

6. for all $x_0 \neq -6, 3$

13. **(a)**

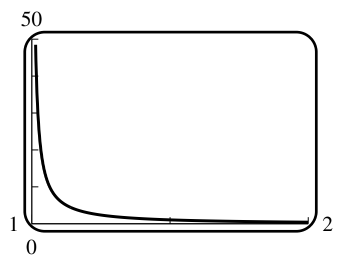
2	1.5	1.1	1.01	1.001	0	0.5	0.9	0.99	0.999
0.1429	0.2105	0.3021	0.3300	0.3330	1.0000	0.5714	0.3690	0.3367	0.3337



The limit is 1/3.

(b)

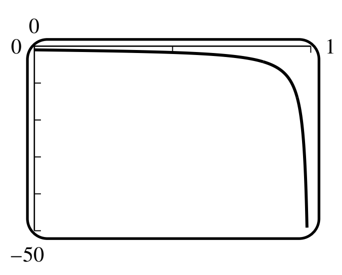
2	1.5	1.1	1.01	1.001	1.0001
0.4286	1.0526	6.344	66.33	666.3	6666.3



The limit is $+\infty$.

(c)

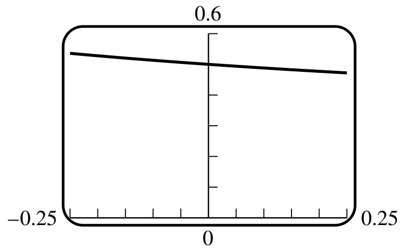
0	0.5	0.9	0.99	0.999	0.9999
-1	-1.7143	-7.0111	-67.001	-667.0	-6667.0



The limit is $-\infty$.

14. (a)

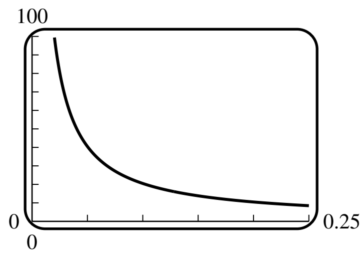
−0.25	−0.1	−0.001	−0.0001	0.0001	0.001	0.1	0.25
0.5359	0.5132	0.5001	0.5000	0.5000	0.4999	0.4881	0.4721



The limit is 1/2.

(b)

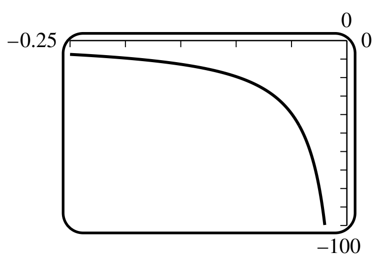
0.25	0.1	0.001	0.0001
8.4721	20.488	2000.5	20001



The limit is $+\infty$.

(c)

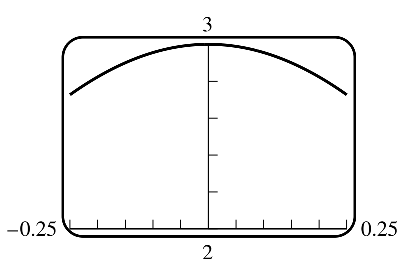
−0.25	−0.1	−0.001	−0.0001
−7.4641	−19.487	−1999.5	−20000



The limit is $-\infty$.

15. (a)

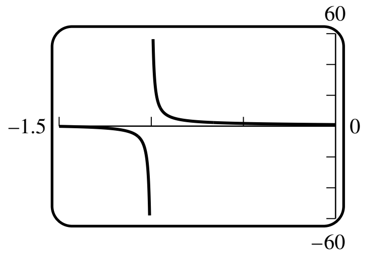
−0.25	−0.1	−0.001	−0.0001	0.0001	0.001	0.1	0.25
2.7266	2.9552	3.0000	3.0000	3.0000	3.0000	2.9552	2.7266



The limit is 3.

(b)

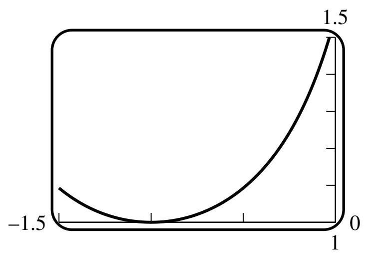
0	−0.5	−0.9	−0.99	−0.999	−1.5	−1.1	−1.01	−1.001
1	1.7552	6.2161	54.87	541.1	−0.1415	−4.536	−53.19	−539.5



The limit does not exist.

16. **(a)**

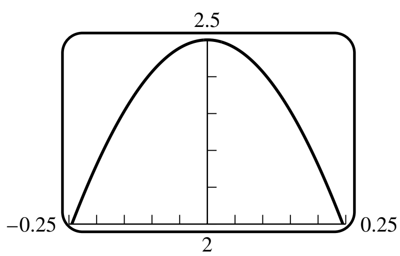
0	−0.5	−0.9	−0.99	−0.999	−1.5	−1.1	−1.01	−1.001
1.5574	1.0926	1.0033	1.0000	1.0000	1.0926	1.0033	1.0000	1.0000



The limit is 1.

(b)

−0.25	−0.1	−0.001	−0.0001	0.0001	0.001	0.1	0.25
1.9794	2.4132	2.5000	2.5000	2.5000	2.5000	2.4132	1.9794



The limit is 5/2.

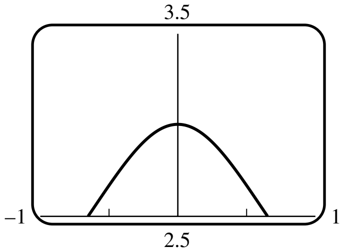
17. $m_{\text{sec}} = \dfrac{x^2-1}{x+1} = x - 1$ which gets close to -2 as x gets close to -1, thus $y - 1 = -2(x+1)$ or $y = -2x - 1$

18. $m_{\text{sec}} = \dfrac{x^2}{x} = x$ which gets close to 0 as x gets close to 0 (doh!), thus $y = 0$

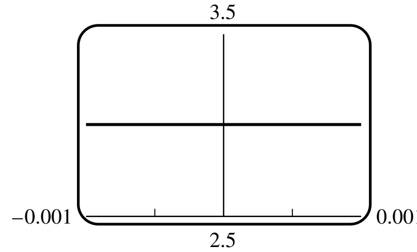
19. $m_{\text{sec}} = \dfrac{x^4-1}{x-1} = x^3 + x^2 + x + 1$ which gets close to 4 as x gets close to 1, thus $y - 1 = 4(x-1)$ or $y = 4x - 3$

20. $m_{\text{sec}} = \dfrac{x^4-1}{x+1} = x^3 - x^2 + x - 1$ which gets close to -4 as x gets close to -1, thus $y - 1 = -4(x+1)$ or $y = -4x - 3$

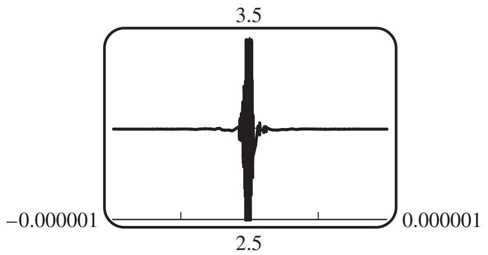
21. **(a)** The limit appears to be 3.



(b) The limit appears to be 3.



(c) The limit does not exist.



22. **(a)**

0.01	0.001	0.0001	0.00001
1.666	0.1667	0.1667	0.17

(b)

0.000001	0.0000001	0.00000001	0.000000001	0.0000000001
0	0	0	0	0

(c) it can be misleading

23. **(a)** The plot over the interval $[-a, a]$ becomes subject to catastrophic subtraction if a is small enough (the size depending on the machine).
(c) It does not.

24. **(a)** the mass of the object while at rest
(b) the limiting mass as the velocity approaches the speed of light; the mass is unbounded

25. **(a)** The length of the rod while at rest
(b) The limit is zero. The length of the rod approaches zero

EXERCISE SET 2.2

1. **(a)** -6 **(b)** 13 **(c)** -8 **(d)** 16 **(e)** 2 **(f)** $-1/2$
(g) The limit doesn't exist because the denominator tends to zero but the numerator doesn't.
(h) The limit doesn't exist because the denominator tends to zero but the numerator doesn't.

2. **(a)** 0
(b) The limit doesn't exist because $\lim f$ doesn't exist and $\lim g$ does.
(c) 0 **(d)** 3 **(e)** 0
(f) The limit doesn't exist because the denominator tends to zero but the numerator doesn't.
(g) The limit doesn't exist because $\sqrt{f(x)}$ is not defined for $0 \le x < 2$.
(h) 1

3. 6 **4.** 27 **5.** 3/4 **6.** -3 **7.** 4 **8.** 12

9. -4/5 **10.** 0 **11.** -3 **12.** 1 **13.** 3/2 **14.** 4/3

15. $+\infty$ **16.** $-\infty$ **17.** does not exist **18.** $+\infty$

19. $-\infty$ **20.** does not exist **21.** $+\infty$ **22.** $-\infty$

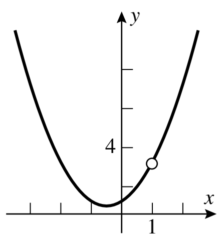
23. does not exist **24.** $-\infty$ **25.** $+\infty$ **26.** does not exist

27. $+\infty$ **28.** $+\infty$ **29.** 6 **30.** 4

31. (a) 2
(b) 2
(c) 2

32. (a) -2
(b) 0
(c) does not exist

33. (a) 3
(b)



34. (a) -6
(b) $F(x) = x - 3$

35. (a) Theorem 2.2.2(a) doesn't apply; moreover one cannot add/subtract infinities.

(b) $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{x^2}\right) = \lim_{x\to 0^+}\left(\frac{x-1}{x^2}\right) = -\infty$

36. $\lim_{x\to 0^-}\left(\frac{1}{x} + \frac{1}{x^2}\right) = \lim_{x\to 0^-}\frac{x+1}{x^2} = +\infty$

37. $\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+4}+2\right)} = \frac{1}{4}$

38. $\lim_{x\to 0}\frac{x^2}{x\left(\sqrt{x+4}+2\right)} = 0$

39. The left and/or right limits could be plus or minus infinity; or the limit could exist, or equal any preassigned real number. For example, let $q(x) = x - x_0$ and let $p(x) = a(x - x_0)^n$ where n takes on the values $0, 1, 2$.

40. (a) If $x < 0$ then $f(x) = \frac{ax + bx - ax + bx}{2x} = b$, so the limit is b.
(b) Similarly if $x > 0$ then $f(x) = a$, so the limit is a.
(c) Since the left limit is a and the right limit is b, the limit can only exist if $a = b$, in which case $f(x) = a$ for all $x \neq 0$ and the limit is a.

EXERCISE SET 2.3

1. (a) $-\infty$
(b) $+\infty$

2. (a) 2
(b) 0

3. (a) 0
(b) -1

4. (a) does not exist
(b) 0

5. **(a)** -12 **(b)** 21 **(c)** -15 **(d)** 25
(e) 2 **(f)** $-3/5$ **(g)** 0
(h) The limit doesn't exist because the denominator tends to zero but the numerator doesn't.

6. **(a)** 20 **(b)** 0 **(c)** $+\infty$ **(d)** $-\infty$
(e) $(-42)^{1/3}$ **(f)** $-6/7$ **(g)** 7 **(h)** $-7/12$

7. $-\infty$ **8.** $+\infty$ **9.** $+\infty$ **10.** $+\infty$ **11.** $3/2$

12. $5/2$ **13.** 0 **14.** 0 **15.** 0 **16.** $5/3$

17. $-5^{1/3}/2$ **18.** $\sqrt[3]{3/2}$ **19.** $-\sqrt{5}$ **20.** $\sqrt{5}$ **21.** $1/\sqrt{6}$

22. $-1/\sqrt{6}$ **23.** $\sqrt{3}$ **24.** $\sqrt{3}$ **25.** $-\infty$ **26.** $+\infty$

27. $-1/7$ **28.** $4/7$

29. It appears that $\lim_{t\to+\infty} n(t) = +\infty$, and $\lim_{t\to+\infty} e(t) = c$.

30. **(a)** It is the initial temperature of the potato (400° F).
(b) It is the ambient temperature, i.e. the temperature of the room.

31. **(a)** $+\infty$ **(b)** -5 **32.** **(a)** 0 **(b)** -6

33. $$\lim_{x\to+\infty}(\sqrt{x^2+3}-x)\frac{\sqrt{x^2+3}+x}{\sqrt{x^2+3}+x} = \lim_{x\to+\infty}\frac{3}{\sqrt{x^2+3}+x} = 0$$

34. $$\lim_{x\to+\infty}(\sqrt{x^2-3x}-x)\frac{\sqrt{x^2-3x}+x}{\sqrt{x^2-3x}+x} = \lim_{x\to+\infty}\frac{-3x}{\sqrt{x^2-3x}+x} = -3/2$$

35. $$\lim_{x\to+\infty}\left(\sqrt{x^2+ax}-x\right)\frac{\sqrt{x^2+ax}+x}{\sqrt{x^2+ax}+x} = \lim_{x\to+\infty}\frac{ax}{\sqrt{x^2+ax}+x} = a/2$$

36. $$\lim_{x\to+\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\frac{\sqrt{x^2+ax}+\sqrt{x^2+bx}}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} = \lim_{x\to+\infty}\frac{(a-b)x}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} = \frac{a-b}{2}$$

37. $\lim_{x\to+\infty} p(x) = (-1)^n\infty$ and $\lim_{x\to-\infty} p(x) = +\infty$

38. If $m > n$ the limits are both zero. If $m = n$ the limits are both 1. If $n > m$ the limits are $(-1)^{n+m}\infty$ and $+\infty$, respectively.

39. If $m > n$ the limits are both zero. If $m = n$ the limits are both equal to a_m, the leading coefficient of p. If $n > m$ the limits are $\pm\infty$ where the sign depends on the sign of a_m and whether n is even or odd.

40. **(a)** $p(x) = q(x) = x$ **(b)** $p(x) = x$, $q(x) = x^2$
(c) $p(x) = x^2$, $q(x) = x$ **(d)** $p(x) = x + 3$, $q(x) = x$

41. If $m > n$ the limit is 0. If $m = n$ the limit is -3. If $m < n$ and $n - m$ is odd, then the limit is $+\infty$; if $m < n$ and $n - m$ is even, then the limit is $-\infty$.

42. If $m > n$ the limit is zero. If $m = n$ the limit is c_m/d_m. If $n > m$ the limit is $+\infty$ if $c_m d_m > 0$ and $-\infty$ if $c_m d_m < 0$.

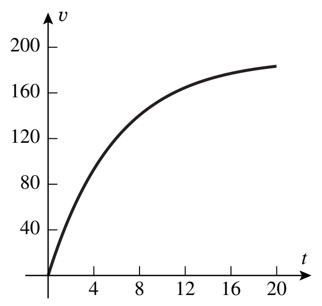
43. $+\infty$ **44.** 0 **45.** $+\infty$ **46.** 0

47. 1 **48.** -1 **49.** 1 **50.** -1

51. $-\infty$ **52.** $-\infty$ **53.** $-\infty$ **54.** $+\infty$

55. 1 **56.** -1 **57.** $+\infty$ **58.** $-\infty$

59. **(a)**

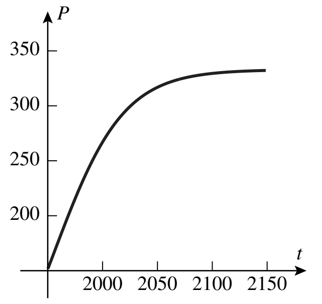


(b) $\lim_{t\to\infty} v = 190\left(1 - \lim_{t\to\infty} e^{-0.168t}\right) = 190$, so the asymptote is $v = c = 190$ ft/sec.

(c) Due to air resistance (and other factors) this is the maximum speed that a sky diver can attain.

60. **(a)** $50371.7/(151.3 + 181.626) \approx 151.3$ million

(b)



(c) $\lim_{t\to\infty} p(t) = \dfrac{50371.7}{151.33 + 181.626 \lim_{t\to\infty} e^{-0.031636(t-1950)}} = \dfrac{50371.7}{151.33} \approx 333$ million

(d) The population becomes stable at this number.

61. $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = L$

62. **(a)** Make the substitution $t = 1/x$ to see that they are equal.

(b) Make the substitution $t = 1/x$ to see that they are equal.

63. $\dfrac{x+1}{x} = 1 + \dfrac{1}{x}$, so $\lim_{x\to+\infty} \dfrac{(x+1)^x}{x^x} = e$ from Figure 1.6.6.

64. If x is large enough, then $1 + \dfrac{1}{x} > 0$, and so $\dfrac{|x+1|^x}{|x|^x} = \left(1 + \dfrac{1}{x}\right)^x$, hence $\lim_{x\to-\infty} \dfrac{|x+1|^x}{|x|^x} = \lim_{x\to-\infty} \left(1 + \dfrac{1}{x}\right)^x = e$ by Figure 1.6.6.

65. Set $t = -x$, then get $\lim_{t\to-\infty} \left(1 + \dfrac{1}{t}\right)^t = e$ by Figure 1.6.6.

66. Set $t = -x$ then $\lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^t = e$

67. Same as $\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^x = \frac{1}{e}$ by Exercise 65.

68. Same as $\lim_{t\to+\infty}\frac{|t-1|^{-t}}{|t|^{-t}} = \frac{1}{e}$ by Exercise 64.

69. $\lim_{x\to+\infty}\left(x+\frac{2}{x}\right)^{3x} \geq \lim_{x\to+\infty} x^{3x}$ which is clearly $+\infty$.

70. If $x \leq -1$ then $\frac{2}{x} \geq -2$, consequently $|x| + \frac{2}{x} \geq |x| - 2$. But $|x| - 2$ gets arbitrarily large, and hence $\left(|x| - \frac{2}{x}\right)^{3x}$ gets arbitrarily small, since the exponent is negative. Thus $\lim_{x\to-\infty}\left(|x|+\frac{2}{x}\right)^{3x} = 0$.

71. Set $t = 1/x$, then $\lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^t = e$

72. Set $t = 1/x$ then $\lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^t = e$

73. Set $t = -1/x$, then $\lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{-t} = \frac{1}{e}$

74. Set $x = -1/t$, then $\lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^{-t} = \frac{1}{e}$.

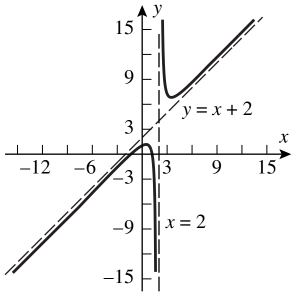
75. Set $t = -1/(2x)$, then $\lim_{t\to+\infty}\left(1-\frac{1}{t}\right)^{-6t} = e^6$

76. Set $t = 1/(2x)$, then $\lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{6t} = e^6$

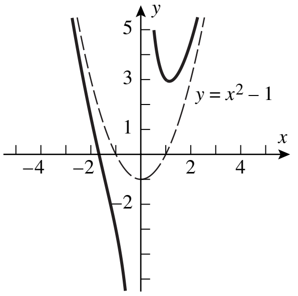
77. Set $t = 1/(2x)$, then $\lim_{t\to-\infty}\left(1-\frac{1}{t}\right)^{6t} = \frac{1}{e^6}$

78. Set $t = 1/(2x)$, then $\lim_{t\to+\infty}\left(1-\frac{1}{t}\right)^{6t} = \frac{1}{e^6}$

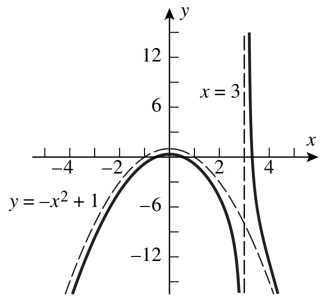
79. $f(x) = x + 2 + \frac{2}{x-2}$,
so $\lim_{x\to\pm\infty}(f(x) - (x+2)) = 0$
and $f(x)$ is asymptotic to $y = x + 2$.



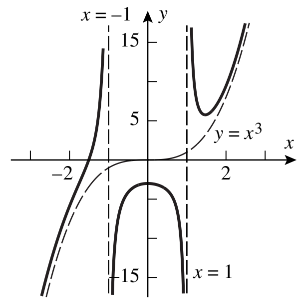
80. $f(x) = x^2 - 1 + 3/x$,
so $\lim_{x\to\pm\infty}[f(x) - (x^2 - 1) = 0]$
and $f(x)$ is asymptotic to $y = x^2 - 1$.



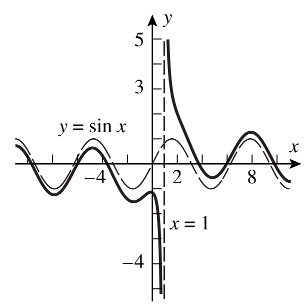
81. $f(x) = -x^2 + 1 + 2/(x-3)$
so $\lim\limits_{x\to\pm\infty} [f(x) - (-x^2+1)] = 0$
and $f(x)$ is asymptotic to $y = -x^2 + 1$.



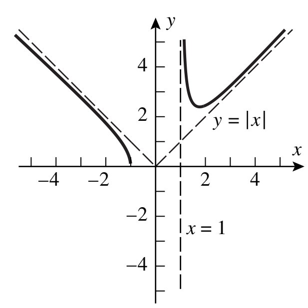
82. $f(x) = x^3 + \dfrac{3}{2(x-1)} - \dfrac{3}{2(x+1)}$
so $\lim\limits_{x\to\pm\infty} [f(x) - x^3] = 0$
and $f(x)$ is asymptotic to $y = x^3$.



83. $f(x) - \sin x = 0$ and $f(x)$ is asymptotic to $y = \sin x$.



84. Note that the function is not defined for $-1 < x \le 1$. For x outside this interval we have $f(x) = \sqrt{x^2 + \dfrac{2}{x-1}}$ which suggests that $\lim\limits_{x\to\pm\infty} [f(x) - |x|\,] = 0$ (this can be checked with a CAS) and hence $f(x)$ is asymptotic to $y = |x|$.



EXERCISE SET 2.4

1. **(a)** $|f(x) - f(0)| = |x + 2 - 2| = |x| < 0.1$ if and only if $|x| < 0.1$

(b) $|f(x) - f(3)| = |(4x - 5) - 7| = 4|x - 3| < 0.1$ if and only if $|x - 3| < (0.1)/4 = 0.0025$

(c) $|f(x) - f(4)| = |x^2 - 16| < \epsilon$ if $|x - 4| < \delta$. We get $f(x) = 16 + \epsilon = 16.001$ at $x = 4.000124998$, which corresponds to $\delta = 0.000124998$; and $f(x) = 16 - \epsilon = 15.999$ at $x = 3.999874998$, for which $\delta = 0.000125002$. Use the smaller δ: thus $|f(x) - 16| < \epsilon$ provided $|x - 4| < 0.000125$ (to six decimals).

2. **(a)** $|f(x) - f(0)| = |2x + 3 - 3| = 2|x| < 0.1$ if and only if $|x| < 0.05$

(b) $|f(x) - f(0)| = |2x + 3 - 3| = 2|x| < 0.01$ if and only if $|x| < 0.005$

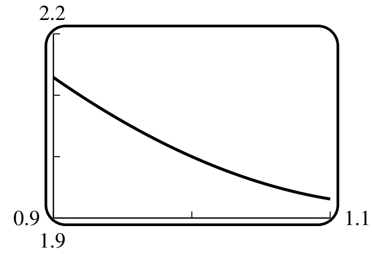
(c) $|f(x) - f(0)| = |2x + 3 - 3| = 2|x| < 0.0012$ if and only if $|x| < 0.0006$

3. **(a)** $x_1 = (1.95)^2 = 3.8025, x_2 = (2.05)^2 = 4.2025$

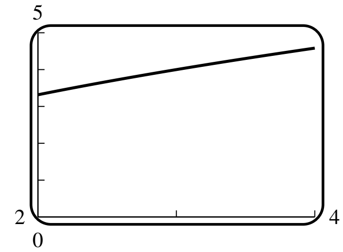
(b) $\delta = \min(\,|4 - 3.8025|, |4 - 4.2025|\,) = 0.1975$

4. **(a)** $x_1 = 1/(1.1) = 0.909090\ldots, x_2 = 1/(0.9) = 1.111111\ldots$
(b) $\delta = \min(\,|1 - 0.909090|, |1 - 1.111111|\,) = 0.0909090\ldots$

5. $|(x^3 - 4x + 5) - 2| < 0.05$, $-0.05 < (x^3 - 4x + 5) - 2 < 0.05$, $1.95 < x^3 - 4x + 5 < 2.05$; $x^3 - 4x + 5 = 1.95$ at $x = 1.0616$, $x^3 - 4x + 5 = 2.05$ at $x = 0.9558$; $\delta = \min(1.0616 - 1, 1 - 0.9558) = 0.0442$



6. $\sqrt{5x+1} = 3.5$ at $x = 2.25$, $\sqrt{5x+1} = 4.5$ at $x = 3.85$, so $\delta = \min(3 - 2.25, 3.85 - 3) = 0.75$



7. With the TRACE feature of a calculator we discover that (to five decimal places) $(0.87000, 1.80274)$ and $(1.13000, 2.19301)$ belong to the graph. Set $x_0 = 0.87$ and $x_1 = 1.13$. Since the graph of $f(x)$ rises from left to right, we see that if $x_0 < x < x_1$ then $1.80274 < f(x) < 2.19301$, and therefore $1.8 < f(x) < 2.2$. So we can take $\delta = 0.13$.

8. From a calculator plot we conjecture that $\lim_{x \to 0} f(x) = 2$. Using the TRACE feature we see that the points $(\pm 0.2, 1.94709)$ belong to the graph. Thus if $-0.2 < x < 0.2$ then $1.95 < f(x) \leq 2$ and hence $|f(x) - L| < 0.05 < 0.1 = \epsilon$.

9. $|2x - 8| = 2|x - 4| < 0.1$ if $|x - 4| < 0.05$, $\delta = 0.05$

10. $|5x - 2 - 13| = 5|x - 3| < 0.01$ if $|x - 3| < 0.002$, $\delta = 0.002$

11. $\left|\dfrac{x^2 - 9}{x - 3} - 6\right| = |x + 3 - 6| = |x - 3| < 0.05$ if $|x - 3| < 0.05, \delta = 0.05$

12. $\left|\dfrac{4x^2 - 1}{2x + 1} + 2\right| = |2x - 1 + 2| = |2x + 1| < 0.05$ if $\left|x + \dfrac{1}{2}\right| < 0.025, \delta = 0.025$

13. On the interval $[1, 3]$ we have $|x^2 + x + 2| \leq 14$, so $|x^3 - 8| = |x - 2||x^2 + x + 2| \leq 14|x - 2| < 0.001$ provided $|x - 2| < 0.001 \cdot \dfrac{1}{14}$; but $0.00005 < \dfrac{0.001}{14}$, so for convenience we take $\delta = 0.00005$ (there is no need to choose an 'optimal' δ).

14. Since $\sqrt{x} > 0, |\sqrt{x} - 2| = \dfrac{|x - 4|}{|\sqrt{x} + 2|} < \dfrac{|x - 4|}{2} < 0.001$ if $|x - 4| < 0.002, \delta = 0.002$

15. if $\delta \leq 1$ then $|x| > 3$, so $\left|\dfrac{1}{x} - \dfrac{1}{5}\right| = \dfrac{|x - 5|}{5|x|} \leq \dfrac{|x - 5|}{15} < 0.05$ if $|x - 5| < 0.75$, $\delta = 3/4$

16. $|x - 0| = |x| < 0.05$ if $|x| < 0.05$, $\delta = 0.05$

17. **(a)** $\lim_{x \to 4} f(x) = 3$

(b) $|10f(x) - 30| = 10|f(x) - 3| < 0.005$ provided $|f(x) - 3| < 0.0005$, which is true for $|x - 3| < 0.0001, \delta = 0.0001$

18. **(a)** $\lim_{x \to 3} f(x) = 7$; $\lim_{x \to 3} g(x) = 5$

(b) $|7f(x) - 21| < 0.03$ is equivalent to $|f(x) - 7| < 0.01$, so let $\epsilon = 0.01$ in condition (i): then when $|x - 3| < \delta = 0.01^2 = 0.0001$, it follows that $|f(x) - 7| < 0.01$, or $|3f(x) - 21| < 0.03$.

19. It suffices to have $|10f(x) + 2x - 38| \le |10f(x) - 30| + 2|x - 4| < 0.01$, by the triangle inequality. To ensure $|10f(x) - 30| < 0.005$ use Exercise 17 (with $\epsilon = 0.0005$) to get $\delta = 0.0001$. Then $|x - 4| < \delta$ yields $|10f(x) + 2x - 38| \le 10|f(x) - 3| + 2|x - 4| \le (10)\dot{0}.0005 + (2)\dot{0}.0001 \le 0.005 + 0.0002 < 0.01$

20. Let $\delta = 0.0009$. By the triangle inequality $|3f(x) + g(x) - 26| \le 3|f(x) - 7| + |g(x) - 5| \le 3 \cdot \sqrt{0.0009} + 0.0072 = 0.03 + 0.0072 < 0.06$.

21. $|3x - 15| = 3|x - 5| < \epsilon$ if $|x - 5| < \frac{1}{3}\epsilon$, $\delta = \frac{1}{3}\epsilon$

22. $|7x + 5 + 2| = 7|x + 1| < \epsilon$ if $|x + 1| < \frac{1}{7}\epsilon$, $\delta = \frac{1}{7}\epsilon$

23. $\left|\dfrac{2x^2 + x}{x} - 1\right| = |2x| < \epsilon$ if $|x| < \frac{1}{2}\epsilon$, $\delta = \frac{1}{2}\epsilon$

24. $\left|\dfrac{x^2 - 9}{x + 3} - (-6)\right| = |x + 3| < \epsilon$ if $|x + 3| < \epsilon$, $\delta = \epsilon$

25. $|f(x) - 3| = |x + 2 - 3| = |x - 1| < \epsilon$ if $0 < |x - 1| < \epsilon$, $\delta = \epsilon$

26. $|9 - 2x - 5| = 2|x - 2| < \epsilon$ if $0 < |x - 2| < \frac{1}{2}\epsilon$, $\delta = \frac{1}{2}\epsilon$

27. **(a)** $|(3x^2 + 2x - 20 - 300| = |3x^2 + 2x - 320| = |(3x + 32)(x - 10)| = |3x + 32| \cdot |x - 10|$

(b) If $|x - 10| < 1$ then $|3x + 32| < 65$, since clearly $x < 11$

(c) $\delta = \min(1, \epsilon/65)$; $\quad |3x + 32| \cdot |x - 10| < 65 \cdot |x - 10| < 65 \cdot \epsilon/65 = \epsilon$

28. **(a)** $\left|\dfrac{28}{3x + 1} - 4\right| = \left|\dfrac{28 - 12x - 4}{3x + 1}\right| = \left|\dfrac{-12x + 24}{3x + 1}\right| = \left|\dfrac{12}{3x + 1}\right| \cdot |x - 2|$

(b) If $|x - 2| < 4$ then $-2 < x < 6$, so x can be very close to $-1/3$, hence $\left|\dfrac{12}{3x + 1}\right|$ is not bounded.

(c) If $|x - 2| < 1$ then $1 < x < 3$ and $3x + 1 > 4$, so $\left|\dfrac{12}{3x + 1}\right| < \dfrac{12}{4} = 3$

(d) $\delta = \min(1, \epsilon/3)$; $\quad \left|\dfrac{12}{3x + 1}\right| \cdot |x - 2| < 3 \cdot |x - 2| < 3 \cdot \epsilon/3 = \epsilon$

29. if $\delta < 1$ then $|2x^2 - 2| = 2|x - 1||x + 1| < 6|x - 1| < \epsilon$ if $|x - 1| < \frac{1}{6}\epsilon$, $\delta = \min(1, \frac{1}{6}\epsilon)$

30. If $\delta < 1$ then $|x^2 + x - 12| = |x + 4| \cdot |x - 3| < 5|x - 3| < \epsilon$ if $|x - 3| < \epsilon/5, \delta = \min(1, \frac{1}{5}\epsilon)$

31. If $\delta < \frac{1}{2}$ and $|x - (-2)| < \delta$ then $-\frac{5}{2} < x < -\frac{3}{2}$, $x + 1 < -\frac{1}{2}$, $|x + 1| > \frac{1}{2}$; then
$\left|\dfrac{1}{x + 1} - (-1)\right| = \dfrac{|x + 2|}{|x + 1|} < 2|x + 2| < \epsilon$ if $|x + 2| < \dfrac{1}{2}\epsilon$, $\delta = \min\left(\dfrac{1}{2}, \dfrac{1}{2}\epsilon\right)$

32. If $\delta < \frac{1}{4}$ then $\left|\dfrac{2x + 3}{x} - 8\right| = \dfrac{|6x - 3|}{|x|} < \dfrac{6|x - \frac{1}{2}|}{\frac{1}{4}} = 24|x - \dfrac{1}{2}| < \epsilon$ if $|x - \frac{1}{2}| < \epsilon/24, \delta = \min(\frac{1}{4}, \frac{\epsilon}{24})$

33. $|\sqrt{x}-2| = \left|(\sqrt{x}-2)\dfrac{\sqrt{x}+2}{\sqrt{x}+2}\right| = \left|\dfrac{x-4}{\sqrt{x}+2}\right| < \dfrac{1}{2}|x-4| < \epsilon$ if $|x-4| < 2\epsilon$, $\delta = 2\epsilon$

34. If $x < 2$ then $|f(x)-5| = |9-2x-5| = 2|x-2| < \epsilon$ if $|x-2| < \frac{1}{2}\epsilon, \delta_1 = \frac{1}{2}\epsilon$. If $x > 2$ then $|f(x)-5| = |3x-1-5| = 3|x-2| < \epsilon$ if $|x-2| < \frac{1}{3}\epsilon, \delta_2 = \frac{1}{3}\epsilon$ Now let $\delta = \min(\delta_1, \delta_2)$ then for any x with $|x-2| < \delta, |f(x)-5| < \epsilon$

35. **(a)** $|f(x)-L| = \dfrac{1}{x^2} < 0.1$ if $x > \sqrt{10}$, $N = \sqrt{10}$

(b) $|f(x)-L| = \left|\dfrac{x}{x+1} - 1\right| = \left|\dfrac{1}{x+1}\right| < 0.01$ if $x+1 > 100$, $N = 99$

(c) $|f(x)-L| = \left|\dfrac{1}{x^3}\right| < \dfrac{1}{1000}$ if $|x| > 10$, $x < -10$, $N = -10$

(d) $|f(x)-L| = \left|\dfrac{x}{x+1} - 1\right| = \left|\dfrac{1}{x+1}\right| < 0.01$ if $|x+1| > 100$, $-x-1 > 100$, $x < -101$, $N = -101$

36. **(a)** $\left|\dfrac{1}{x^3}\right| < 0.1$, $x > 10^{1/3}$, $N = 10^{1/3}$ **(b)** $\left|\dfrac{1}{x^3}\right| < 0.01$, $x > 100^{1/3}$, $N = 100^{1/3}$

(c) $\left|\dfrac{1}{x^3}\right| < 0.001$, $x > 10$, $N = 10$

37. **(a)** $\dfrac{x_1^2}{1+x_1^2} = 1-\epsilon$, $x_1 = -\sqrt{\dfrac{1-\epsilon}{\epsilon}}$; $\dfrac{x_2^2}{1+x_2^2} = 1-\epsilon$, $x_2 = \sqrt{\dfrac{1-\epsilon}{\epsilon}}$

(b) $N = \sqrt{\dfrac{1-\epsilon}{\epsilon}}$ **(c)** $N = -\sqrt{\dfrac{1-\epsilon}{\epsilon}}$

38. **(a)** $x_1 = -1/\epsilon^3$; $x_2 = 1/\epsilon^3$ **(b)** $N = 1/\epsilon^3$ **(c)** $N = -1/\epsilon^3$

39. $\dfrac{1}{x^2} < 0.01$ if $|x| > 10$, $N = 10$

40. $\dfrac{1}{x+2} < 0.005$ if $|x+2| > 200$, $x > 198$, $N = 198$

41. $\left|\dfrac{x}{x+1} - 1\right| = \left|\dfrac{1}{x+1}\right| < 0.001$ if $|x+1| > 1000$, $x > 999$, $N = 999$

42. $\left|\dfrac{4x-1}{2x+5} - 2\right| = \left|\dfrac{11}{2x+5}\right| < 0.1$ if $|2x+5| > 110$, $2x > 105$, $N = 52.5$

43. $\left|\dfrac{1}{x+2} - 0\right| < 0.005$ if $|x+2| > 200$, $-x-2 > 200$, $x < -202$, $N = -202$

44. $\left|\dfrac{1}{x^2}\right| < 0.01$ if $|x| > 10$, $-x > 10$, $x < -10$, $N = -10$

45. $\left|\dfrac{4x-1}{2x+5} - 2\right| = \left|\dfrac{11}{2x+5}\right| < 0.1$ if $|2x+5| > 110$, $-2x-5 > 110$, $2x < -115$, $x < -57.5$, $N = -57.5$

46. $\left|\dfrac{x}{x+1} - 1\right| = \left|\dfrac{1}{x+1}\right| < 0.001$ if $|x+1| > 1000$, $-x-1 > 1000$, $x < -1001$, $N = -1001$

47. $\left|\frac{1}{x^2}\right| < \epsilon$ if $|x| > \frac{1}{\sqrt{\epsilon}}$, $N = \frac{1}{\sqrt{\epsilon}}$

48. $\left|\frac{1}{x+2}\right| < \epsilon$ if $|x+2| > \frac{1}{\epsilon}$, $x+2 > \frac{1}{\epsilon}$, $x > \frac{1}{\epsilon} - 2$, $N = \frac{1}{\epsilon} - 2$

49. $\left|\frac{4x-1}{2x+5} - 2\right| = \left|\frac{11}{2x+5}\right| < \epsilon$ if $|2x+5| > \frac{11}{\epsilon}$, $-2x-5 > \frac{11}{\epsilon}$, $2x < -\frac{11}{\epsilon} - 5$, $x < -\frac{11}{2\epsilon} - \frac{5}{2}$, $N = -\frac{5}{2} - \frac{11}{2\epsilon}$

50. $\left|\frac{x}{x+1} - 1\right| = \left|\frac{1}{x+1}\right| < \epsilon$ if $|x+1| > \frac{1}{\epsilon}$, $-x-1 > \frac{1}{\epsilon}$, $x < -1 - \frac{1}{\epsilon}$, $N = -1 - \frac{1}{\epsilon}$

51. $\left|\frac{2\sqrt{x}}{\sqrt{x}-1} - 2\right| = \left|\frac{2}{\sqrt{x}-1}\right| < \epsilon$ if $\sqrt{x} - 1 > \frac{2}{\epsilon}$, $\sqrt{x} > 1 + \frac{2}{\epsilon}$, $x > \left(1 + \frac{2}{\epsilon}\right)^2$, $N > \left(1 + \frac{2}{\epsilon}\right)^2$

52. $2^x < \epsilon$ if $x < \log_2 \epsilon$, $N < \log_2 \epsilon$

53. **(a)** $\frac{1}{x^2} > 100$ if $|x| < \frac{1}{10}$

(b) $\frac{1}{|x-1|} > 1000$ if $|x-1| < \frac{1}{1000}$

(c) $\frac{-1}{(x-3)^2} < -1000$ if $|x-3| < \frac{1}{10\sqrt{10}}$

(d) $-\frac{1}{x^4} < -10000$ if $x^4 < \frac{1}{10000}$, $|x| < \frac{1}{10}$

54. **(a)** $\frac{1}{(x-1)^2} > 10$ if and only if $|x-1| < \frac{1}{\sqrt{10}}$

(b) $\frac{1}{(x-1)^2} > 1000$ if and only if $|x-1| < \frac{1}{10\sqrt{10}}$

(c) $\frac{1}{(x-1)^2} > 100000$ if and only if $|x-1| < \frac{1}{100\sqrt{10}}$

55. if $M > 0$ then $\frac{1}{(x-3)^2} > M$, $0 < (x-3)^2 < \frac{1}{M}$, $0 < |x-3| < \frac{1}{\sqrt{M}}$, $\delta = \frac{1}{\sqrt{M}}$

56. if $M < 0$ then $\frac{-1}{(x-3)^2} < M$, $0 < (x-3)^2 < -\frac{1}{M}$, $0 < |x-3| < \frac{1}{\sqrt{-M}}$, $\delta = \frac{1}{\sqrt{-M}}$

57. if $M > 0$ then $\frac{1}{|x|} > M$, $0 < |x| < \frac{1}{M}$, $\delta = \frac{1}{M}$

58. if $M > 0$ then $\frac{1}{|x-1|} > M$, $0 < |x-1| < \frac{1}{M}$, $\delta = \frac{1}{M}$

59. if $M < 0$ then $-\frac{1}{x^4} < M$, $0 < x^4 < -\frac{1}{M}$, $|x| < \frac{1}{(-M)^{1/4}}$, $\delta = \frac{1}{(-M)^{1/4}}$

60. if $M > 0$ then $\frac{1}{x^4} > M$, $0 < x^4 < \frac{1}{M}$, $x < \frac{1}{M^{1/4}}$, $\delta = \frac{1}{M^{1/4}}$

61. if $x > 2$ then $|x+1-3| = |x-2| = x-2 < \epsilon$ if $2 < x < 2+\epsilon$, $\delta = \epsilon$

62. if $x < 1$ then $|3x+2-5| = |3x-3| = 3|x-1| = 3(1-x) < \epsilon$ if $1-x < \frac{1}{3}\epsilon$, $1 - \frac{1}{3}\epsilon < x < 1$, $\delta = \frac{1}{3}\epsilon$

63. if $x > 4$ then $\sqrt{x-4} < \epsilon$ if $x - 4 < \epsilon^2$, $4 < x < 4 + \epsilon^2$, $\delta = \epsilon^2$

64. if $x < 0$ then $\sqrt{-x} < \epsilon$ if $-x < \epsilon^2$, $-\epsilon^2 < x < 0$, $\delta = \epsilon^2$

65. if $x > 2$ then $|f(x) - 2| = |x - 2| = x - 2 < \epsilon$ if $2 < x < 2 + \epsilon$, $\delta = \epsilon$

66. if $x < 2$ then $|f(x) - 6| = |3x - 6| = 3|x - 2| = 3(2 - x) < \epsilon$ if $2 - x < \frac{1}{3}\epsilon$, $2 - \frac{1}{3}\epsilon < x < 2$, $\delta = \frac{1}{3}\epsilon$

67. **(a)** if $M < 0$ and $x > 1$ then $\dfrac{1}{1-x} < M$, $x - 1 < -\dfrac{1}{M}$, $1 < x < 1 - \dfrac{1}{M}$, $\delta = -\dfrac{1}{M}$

(b) if $M > 0$ and $x < 1$ then $\dfrac{1}{1-x} > M$, $1 - x < \dfrac{1}{M}$, $1 - \dfrac{1}{M} < x < 1$, $\delta = \dfrac{1}{M}$

68. **(a)** if $M > 0$ and $x > 0$ then $\dfrac{1}{x} > M$, $x < \dfrac{1}{M}$, $0 < x < \dfrac{1}{M}$, $\delta = \dfrac{1}{M}$

(b) if $M < 0$ and $x < 0$ then $\dfrac{1}{x} < M$, $-x < -\dfrac{1}{M}$, $\dfrac{1}{M} < x < 0$, $\delta = -\dfrac{1}{M}$

69. **(a)** Given any $M > 0$ there corresponds $N > 0$ such that if $x > N$ then $f(x) > M$, $x + 1 > M$, $x > M - 1$, $N = M - 1$.

(b) Given any $M < 0$ there corresponds $N < 0$ such that if $x < N$ then $f(x) < M$, $x + 1 < M$, $x < M - 1$, $N = M - 1$.

70. **(a)** Given any $M > 0$ there corresponds $N > 0$ such that if $x > N$ then $f(x) > M$, $x^2 - 3 > M$, $x > \sqrt{M+3}$, $N = \sqrt{M+3}$.

(b) Given any $M < 0$ there corresponds $N < 0$ such that if $x < N$ then $f(x) < M$, $x^3 + 5 < M$, $x < (M - 5)^{1/3}$, $N = (M - 5)^{1/3}$.

71. if $\delta \le 2$ then $|x - 3| < 2$, $-2 < x - 3 < 2$, $1 < x < 5$, and $|x^2 - 9| = |x + 3||x - 3| < 8|x - 3| < \epsilon$ if $|x - 3| < \frac{1}{8}\epsilon$, $\delta = \min\left(2, \frac{1}{8}\epsilon\right)$

72. **(a)** We don't care about the value of f at $x = a$, because the limit is only concerned with values of x near a. The condition that f be defined for all x (except possibly $x = a$) is necessary, because if some points were excluded then the limit may not exist; for example, let $f(x) = x$ if $1/x$ is not an integer and $f(1/n) = 6$. Then $\lim_{x\to 0} f(x)$ does not exist but it would if the points $1/n$ were excluded.

(b) if $x < 0$ then $\sqrt{x}$ is not defined **(c)** yes; if $\delta \le 0.01$ then $x > 0$, so $\sqrt{x}$ is defined

73. **(a)** 0.4 amperes **(b)** $[0.3947, 0.4054]$ **(c)** $\left[\dfrac{3}{7.5+\delta}, \dfrac{3}{7.5-\delta}\right]$

(d) 0.0187 **(e)** It becomes infinite.

EXERCISE SET 2.5

1. **(a)** no, $x = 2$ **(b)** no, $x = 2$ **(c)** no, $x = 2$ **(d)** yes
(e) yes **(f)** yes

2. **(a)** no, $x = 2$ **(b)** no, $x = 2$ **(c)** no, $x = 2$ **(d)** yes
(e) no, $x = 2$ **(f)** yes

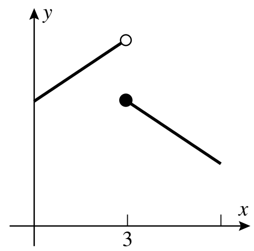
3. **(a)** no, $x = 1, 3$ **(b)** yes **(c)** no, $x = 1$ **(d)** yes
(e) no, $x = 3$ **(f)** yes

4. **(a)** no, $x = 3$ **(b)** yes **(c)** yes **(d)** yes
(e) no, $x = 3$ **(f)** yes

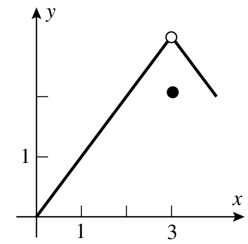
5. **(a)** 3 **(b)** 3

6. $-2/5$

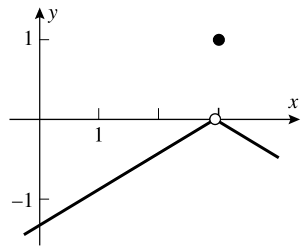
7. **(a)**



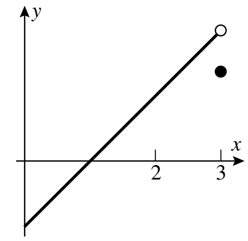
(b)



(c)

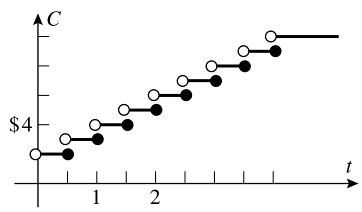


(d)



8. $f(x) = 1/x$, $g(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \sin \dfrac{1}{x} & \text{if} \quad x \neq 0 \end{cases}$

9. **(a)**



(b) One second could cost you one dollar.

10. **(a)** no; disasters (war, flood, famine, pestilence, for example) can cause discontinuities
(b) continuous
(c) not usually continuous; see Exercise 9
(d) continuous

11. none **12.** none **13.** none **14.** $x = -2, 2$ **15.** $x = 0, -1/2$

16. none **17.** $x = -1, 0, 1$ **18.** $x = -4, 0$ **19.** none **20.** $x = -1, 0$

21. none; $f(x) = 2x + 3$ is continuous on $x < 4$ and $f(x) = 7 + \dfrac{16}{x}$ is continuous on $4 < x$; $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = f(4) = 11$ so f is continuous at $x = 4$

22. $\lim_{x \to 1} f(x)$ does not exist so f is discontinuous at $x = 1$

23. **(a)** f is continuous for $x < 1$, and for $x > 1$; $\lim_{x \to 1^-} f(x) = 5$, $\lim_{x \to 1^+} f(x) = k$, so if $k = 5$ then f is continuous for all x

(b) f is continuous for $x < 2$, and for $x > 2$; $\lim_{x \to 2^-} f(x) = 4k$, $\lim_{x \to 2^+} f(x) = 4 + k$, so if $4k = 4 + k$, $k = 4/3$ then f is continuous for all x

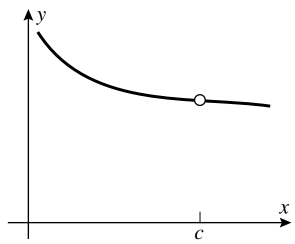
24. **(a)** f is continuous for $x < 3$, and for $x > 3$; $\lim_{x\to 3^-} f(x) = k/9$, $\lim_{x\to 3^+} f(x) = 0$, so if $k = 0$ then f is continuous for all x

(b) f is continuous for $x < 0$, and for $x > 0$; $\lim_{x\to 0^-} f(x)$ doesn't exist unless $k = 0$, and if so then $\lim_{x\to 0^-} f(x) = +\infty$; $\lim_{x\to 0^+} f(x) = 9$, so no value of k

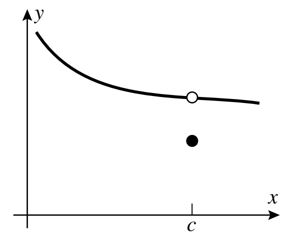
25. f is continuous for $x < -1$, $-1 < x < 2$ and $x > 2$; $\lim_{x\to -1^-} f(x) = 4$, $\lim_{x\to -1^+} f(x) = k$, so $k = 4$ is required. Next, $\lim_{x\to 2^-} f(x) = 3m + k = 3m + 4$, $\lim_{x\to 2^+} f(x) = 9$, so $3m + 4 = 9, m = 5/3$ and f is continuous everywhere if $k = 4, m = 5/3$

26. **(a)** no, f is not defined at $x = 2$ **(b)** no, f is not defined for $x \le 2$
(c) yes **(d)** no, f is not defined for $x \le 2$

27. **(a)**

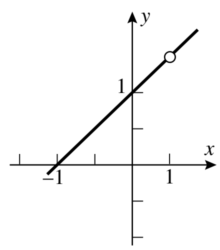


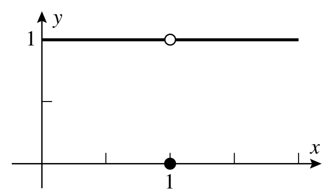
(b)



28. **(a)** $f(c) = \lim_{x\to c} f(x)$

(b) $\lim_{x\to 1} f(x) = 2$ $\lim_{x\to 1} g(x) = 1$





(c) Define $f(1) = 2$ and redefine $g(1) = 1$.

29. **(a)** $x = 0$, $\lim_{x\to 0^-} f(x) = -1 \ne +1 = \lim_{x\to 0^+} f(x)$ so the discontinuity is not removable

(b) $x = -3$; define $f(-3) = -3 = \lim_{x\to -3} f(x)$, then the discontinuity is removable

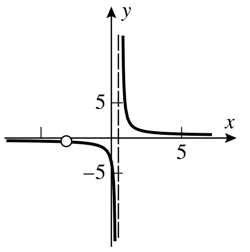
(c) f is undefined at $x = \pm 2$; at $x = 2$, $\lim_{x\to 2} f(x) = 1$, so define $f(2) = 1$ and f becomes continuous there; at $x = -2$, $\lim_{x\to -2}$ does not exist, so the discontinuity is not removable

30. **(a)** f is not defined at $x = 2$; $\lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{x+2}{x^2+2x+4} = \dfrac{1}{3}$, so define $f(2) = \dfrac{1}{3}$ and f becomes continuous there

(b) $\lim_{x\to 2^-} f(x) = 1 \ne 4 = \lim_{x\to 2^+} f(x)$, so f has a nonremovable discontinuity at $x = 2$

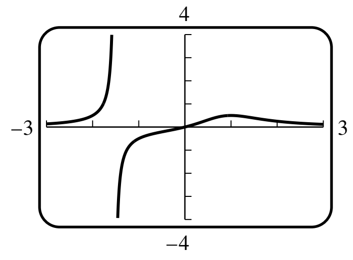
(c) $\lim_{x\to 1} f(x) = 8 \ne f(1)$, so f has a removable discontinuity at $x = 1$

31. **(a)** discontinuity at $x = 1/2$, not removable; at $x = -3$, removable



(b) $2x^2 + 5x - 3 = (2x - 1)(x + 3)$

32. **(a)** there appears to be one discontinuity near $x = -1.52$



(b) one discontinuity at $x \approx -1.52$

33. For $x > 0$, $f(x) = x^{3/5} = (x^3)^{1/5}$ is the composition (Theorem 2.5.6) of the two continuous functions $g(x) = x^3$ and $h(x) = x^{1/5}$ and is thus continuous. For $x < 0$, $f(x) = f(-x)$ which is the composition of the continuous functions $f(x)$ (for positive x) and the continuous function $y = -x$. Hence $f(-x)$ is continuous for all $x > 0$. At $x = 0$, $f(0) = \lim_{x \to 0} f(x) = 0$.

34. $x^4 + 7x^2 + 1 \geq 1 > 0$, thus $f(x)$ is the composition of the polynomial $x^4 + 7x^2 + 1$, the square root $\sqrt{x}$, and the function $1/x$ and is therefore continuous by Theorem 2.5.6.

35. **(a)** Let $f(x) = k$ for $x \neq c$ and $f(c) = 0$; $g(x) = l$ for $x \neq c$ and $g(c) = 0$. If $k = -l$ then $f + g$ is continuous; otherwise it's not.

(b) $f(x) = k$ for $x \neq c$, $f(c) = 1$; $g(x) = l \neq 0$ for $x \neq c$, $g(c) = 1$. If $kl = 1$, then fg is continuous; otherwise it's not.

36. A rational function is the quotient $f(x)/g(x)$ of two polynomials $f(x)$ and $g(x)$. By Theorem 2.5.2 f and g are continuous everywhere; by Theorem 2.5.3 f/g is continuous except when $g(x) = 0$.

37. Since f and g are continuous at $x = c$ we know that $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$. In the following we use Theorem 2.2.2.

(a) $f(c) + g(c) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = \lim_{x \to c}(f(x) + g(x))$ so $f + g$ is continuous at $x = c$.

(b) same as (a) except the $+$ sign becomes a $-$ sign

(c) $f(c) \cdot g(c) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right) = \lim_{x \to c} f(x) \cdot g(x)$ so $f \cdot g$ is continuous at $x = c$

38. $h(x) = f(x) - g(x)$ satisfies $h(a) > 0$, $h(b) < 0$. Use the Intermediate Value Theorem or Theorem 2.5.8.

39. Of course such a function must be discontinuous. Let $f(x) = 1$ on $0 \leq x < 1$, and $f(x) = -1$ on $1 \leq x \leq 2$.

40. **(a)** (i) no (ii) yes **(b)** (i) no (ii) no **(c)** (i) no (ii) no

41. The cone has volume $\pi r^2 h/3$. The function $V(r) = \pi r^2 h$ (for variable r and fixed h) gives the volume of a right circular cylinder of height h and radius r, and satisfies $V(0) < \pi r^2 h/3 < V(r)$. By the Intermediate Value Theorem there is a value c between 0 and r such that $V(c) = \pi r^2 h/3$, so the cylinder of radius c (and height h) has volume equal to that of the cone.

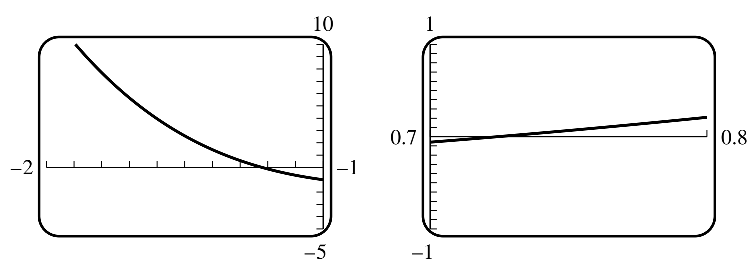
42. A square whose diagonal has length r has area $f(r) = r^2/2$. Note that $f(r) = r^2/2 < \pi r^2/2 < 2r^2 = f(2r)$. By the Intermediate Value Theorem there must be a value c between r and $2r$ such that $f(c) = \pi r^2/2$, i.e. a square of diagonal c whose area is $\pi r^2/2$.

43. If $f(x) = x^3 + x^2 - 2x$ then $f(-1) = 2$, $f(1) = 0$. Use the Intermediate Value Theorem.

44. Since $\lim_{x\to-\infty} p(x) = -\infty$ and $\lim_{x\to+\infty} p(x) = +\infty$ (or vice versa, if the leading coefficient of p is negative), it follows that for $M = -1$ there corresponds $N_1 < 0$, and for $M = 1$ there is $N_2 > 0$, such that $p(x) < -1$ for $x < N_1$ and $p(x) > 1$ for $x > N_2$. Choose $x_1 < N_1$ and $x_2 > N_2$ and use Theorem 2.5.8 on the interval $[x_1, x_2]$ to find a solution of $p(x) = 0$.

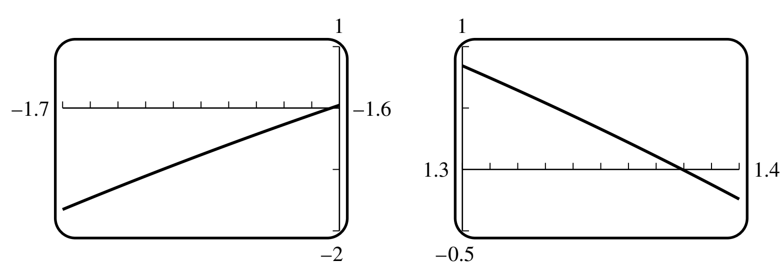
45. For the negative root, use intervals on the x-axis as follows: $[-2, -1]$; since $f(-1.3) < 0$ and $f(-1.2) > 0$, the midpoint $x = -1.25$ of $[-1.3, -1.2]$ is the required approximation of the root. For the positive root use the interval $[0, 1]$; since $f(0.7) < 0$ and $f(0.8) > 0$, the midpoint $x = 0.75$ of $[0.7, 0.8]$ is the required approximation.

46. $x = -1.22$ and $x = 0.72$.



47. For the negative root, use intervals on the x-axis as follows: $[-2, -1]$; since $f(-1.7) < 0$ and $f(-1.6) > 0$, use the interval $[-1.7, -1.6]$. Since $f(-1.61) < 0$ and $f(-1.60) > 0$ the midpoint $x = -1.605$ of $[-1.61, -1.60]$ is the required approximation of the root. For the positive root use the interval $[1, 2]$; since $f(1.3) > 0$ and $f(1.4) < 0$, use the interval $[1.3, 1.4]$. Since $f(1.37) > 0$ and $f(1.38) < 0$, the midpoint $x = 1.375$ of $[1.37, 1.38]$ is the required approximation.

48. $x = -1.603$ and $x = 1.3799$.



49. $x = 2.24$

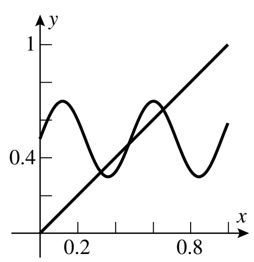
50. Set $f(x) = \dfrac{a}{x-1} + \dfrac{b}{x-3}$. Since $\lim_{x\to 1^+} f(x) = +\infty$ and $\lim_{x\to 3^-} f(x) = -\infty$ there exist $x_1 > 1$ and $x_2 < 3$ (with $x_2 > x_1$) such that $f(x) > 1$ for $1 < x < x_1$ and $f(x) < -1$ for $x_2 < x < 3$. Choose x_3 in $(1, x_1)$ and x_4 in $(x_2, 3)$ and apply Theorem 2.5.8 on $[x_3, x_4]$.

51. The uncoated sphere has volume $4\pi(x-1)^3/3$ and the coated sphere has volume $4\pi x^3/3$. If the volume of the uncoated sphere and of the coating itself are the same, then the coated sphere has twice the volume of the uncoated sphere. Thus $2(4\pi(x-1)^3/3) = 4\pi x^3/3$, or $x^3 - 6x^2 + 6x - 2 = 0$, with the solution $x = 4.847$ cm.

52. Let $g(t)$ denote the altitude of the monk at time t measured in hours from noon of day one, and let $f(t)$ denote the altitude of the monk at time t measured in hours from noon of day two. Then $g(0) < f(0)$ and $g(12) > f(12)$. Use Exercise 38.

53. We must show $\lim_{x\to c} f(x) = f(c)$. Let $\epsilon > 0$; then there exists $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$. But this certainly satisfies Definition 2.4.1.

54. **(a)**



(b) Let $g(x) = x - f(x)$. Then $g(1) \ge 0$ and $g(0) \le 0$; by the Intermediate Value Theorem there is a solution c in $[0, 1]$ of $g(c) = 0$.

EXERCISE SET 2.6

1. none

2. $x = \pi$

3. $n\pi, n = 0, \pm 1, \pm 2, \ldots$

4. $x = n\pi + \pi/2,\ n = 0, \pm 1, \pm 2, \ldots$

5. $x = n\pi,\ n = 0, \pm 1, \pm 2, \ldots$

6. none

7. $2n\pi + \pi/6, 2n\pi + 5\pi/6,\ n = 0, \pm 1, \pm 2, \ldots$

8. $x = n\pi + \pi/2,\ n = 0, \pm 1, \pm 2, \ldots$

9. $[-1, 1]$

10. $(-\infty, -1] \cup [1, \infty)$

11. $(0, 3) \cup (3, +\infty)$

12. $(-\infty, 0) \cup (0, +\infty)$, and if f is defined to be e at $x = 0$, then continuous for all x

13. $(-\infty, -1] \cup [1, \infty)$

14. $(-3, 0) \cup (0, \infty)$

15. **(a)** $\sin x, x^3 + 7x + 1$ **(b)** $|x|, \sin x$ **(c)** $x^3, \cos x, x + 1$
(d) $\sqrt{x}, 3 + x, \sin x, 2x$ **(e)** $\sin x, \sin x$ **(f)** $x^5 - 2x^3 + 1, \cos x$

16. **(a)** Use Theorem 2.5.6. **(b)** $g(x) = \cos x,\ g(x) = \dfrac{1}{x^2+1},\ g(x) = x^2 + 1$

17. $\cos\left(\lim_{x\to+\infty} \dfrac{1}{x}\right) = \cos 0 = 1$

18. $\sin\left(\lim_{x\to+\infty} \dfrac{\pi x}{2 - 3x}\right) = \sin\left(-\dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

19. $\sin^{-1}\left(\lim_{x\to+\infty} \dfrac{x}{1 - 2x}\right) = \sin^{-1}\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$

20. $\ln\left(\lim_{x\to+\infty}\frac{x+1}{x}\right)=\ln(1)=0$

21. $3\lim_{\theta\to 0}\frac{\sin 3\theta}{3\theta}=3$

22. $\frac{1}{2}\lim_{h\to 0}\frac{\sin h}{h}=\frac{1}{2}$

23. $\left(\lim_{\theta\to 0^+}\frac{1}{\theta}\right)\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=+\infty$

24. $\left(\lim_{\theta\to 0}\sin\theta\right)\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=0$

25. $\frac{\tan 7x}{\sin 3x}=\frac{7}{3\cos 7x}\frac{\sin 7x}{7x}\frac{3x}{\sin 3x}$ so $\lim_{x\to 0}\frac{\tan 7x}{\sin 3x}=\frac{7}{3(1)}(1)(1)=\frac{7}{3}$

26. $\frac{\sin 6x}{\sin 8x}=\frac{6}{8}\frac{\sin 6x}{6x}\frac{8x}{\sin 8x}$, so $\lim_{x\to 0}\frac{\sin 6x}{\sin 8x}=\frac{6}{8}=\frac{3}{4}$

27. $\frac{1}{5}\lim_{x\to 0^+}\sqrt{x}\lim_{x\to 0^+}\frac{\sin x}{x}=0$

28. $\frac{1}{3}\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2=\frac{1}{3}$

29. $\left(\lim_{x\to 0}x\right)\left(\lim_{x\to 0}\frac{\sin x^2}{x^2}\right)=0$

30. $\frac{\sin h}{1-\cos h}=\frac{\sin h}{1-\cos h}\frac{1+\cos h}{1+\cos h}=\frac{\sin h(1+\cos h)}{1-\cos^2 h}=\frac{1+\cos h}{\sin h}$; no limit

31. $\frac{t^2}{1-\cos^2 t}=\left(\frac{t}{\sin t}\right)^2$, so $\lim_{t\to 0}\frac{t^2}{1-\cos^2 t}=1$

32. $\cos(\frac{1}{2}\pi-x)=\sin(\frac{1}{2}\pi)\sin x=\sin x$, so $\lim_{x\to 0}\frac{x}{\cos\left(\frac{1}{2}\pi-x\right)}=1$

33. $\frac{\theta^2}{1-\cos\theta}\frac{1+\cos\theta}{1+\cos\theta}=\frac{\theta^2(1+\cos\theta)}{1-\cos^2\theta}=\left(\frac{\theta}{\sin\theta}\right)^2(1+\cos\theta)$ so $\lim_{\theta\to 0}\frac{\theta^2}{1-\cos\theta}=(1)^2 2=2$

34. $\frac{1-\cos 3h}{\cos^2 5h-1}\frac{1+\cos 3h}{1+\cos 3h}=\frac{\sin^2 3h}{-\sin^2 5h}\frac{1}{1+\cos 3h}$, so

$\lim_{x\to 0}\frac{1-\cos 3h}{\cos^2 5h-1}=\lim_{x\to 0}\frac{\sin^2 3h}{-\sin^2 5h}\frac{1}{1+\cos 3h}=-\left(\frac{3}{5}\right)^2\frac{1}{2}=-\frac{9}{50}$

35. $\lim_{x\to 0^+}\sin\left(\frac{1}{x}\right)=\lim_{t\to+\infty}\sin t$; limit does not exist

36. $\lim_{x\to 0}x-3\lim_{x\to 0}\frac{\sin x}{x}=-3$

37. $\frac{2-\cos 3x-\cos 4x}{x}=\frac{1-\cos 3x}{x}+\frac{1-\cos 4x}{x}$. Note that

$\frac{1-\cos 3x}{x}=\frac{1-\cos 3x}{x}\frac{1+\cos 3x}{1+\cos 3x}=\frac{\sin^2 3x}{x(1+\cos 3x)}=\frac{\sin 3x}{x}\frac{\sin 3x}{1+\cos 3x}$. Thus

$\lim_{x\to 0}\frac{2-\cos 3x-\cos 4x}{x}=\lim_{x\to 0}\frac{\sin 3x}{x}\frac{\sin 3x}{1+\cos 3x}+\lim_{x\to 0}\frac{\sin 4x}{x}\frac{\sin 4x}{1+\cos 4x}=0+0=0$

38. $\dfrac{\tan 3x^2 + \sin^2 5x}{x^2} = \dfrac{3}{\cos 3x^2}\dfrac{\sin 3x^2}{3x^2} + 5^2\dfrac{\sin^2 5x}{(5x)^2}$, so

$$\text{limit} = \lim_{x\to 0}\frac{3}{\cos 3x^2}\lim_{x\to 0}\frac{\sin 3x^2}{3x^2} + 25\lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)^2 = 3 + 25 = 28$$

39. a/b

40. k^2s

41.

5.1	5.01	5.001	5.0001	5.00001	4.9	4.99	4.999	4.9999	4.99999
0.098845	0.099898	0.99990	0.099999	0.100000	0.10084	0.10010	0.10001	0.10000	0.10000

The limit is 0.1.

42.

2.1	2.01	2.001	2.0001	2.00001	1.9	1.99	1.999	1.9999	1.99999
0.484559	0.498720	0.499875	0.499987	0.499999	0.509409	0.501220	0.500125	0.500012	0.500001

The limit is 0.5.

43.

−1.9	−1.99	−1.999	−1.9999	−1.99999	−2.1	−2.01	−2.001	−2.0001	−2.00001
−0.898785	−0.989984	−0.999000	−0.999900	−0.999990	−1.097783	−1.009983	−1.001000	−1.000100	−1.000010

The limit is −1.

44.

−0.9	−0.99	−0.999	−0.9999	−0.99999	−1.1	−1.01	−1.001	−1.0001	−1.00001
0.405086	0.340050	0.334001	0.333400	0.333340	0.271536	0.326717	0.332667	0.333267	0.333327

The limit is 1/3.

45. Since $\lim\limits_{x\to 0}\sin(1/x)$ does not exist, no conclusions can be drawn.

46. $k = f(0) = \lim\limits_{x\to 0}\dfrac{\sin 3x}{x} = 3\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x} = 3$, so $k = 3$

47. $\lim\limits_{x\to 0^-} f(x) = k\lim\limits_{x\to 0}\dfrac{\sin kx}{kx\cos kx} = k$, $\lim\limits_{x\to 0^+} f(x) = 2k^2$, so $k = 2k^2$, $k = \dfrac{1}{2}$

48. No; $\sin x/|x|$ has unequal one-sided limits.

49. **(a)** $\lim\limits_{t\to 0^+}\dfrac{\sin t}{t} = 1$

(b) $\lim\limits_{t\to 0^-}\dfrac{1-\cos t}{t} = 0$ (Theorem 2.6.4)

(c) $\sin(\pi - t) = \sin t$, so $\lim\limits_{x\to\pi}\dfrac{\pi - x}{\sin x} = \lim\limits_{t\to 0}\dfrac{t}{\sin t} = 1$

50. $\cos\left(\dfrac{\pi}{2} - t\right) = \sin t$, so $\lim\limits_{x\to 2}\dfrac{\cos(\pi/x)}{x-2} = \lim\limits_{t\to 0}\dfrac{(\pi - 2t)\sin t}{4t} = \lim\limits_{t\to 0}\dfrac{\pi - 2t}{4}\lim\limits_{t\to 0}\dfrac{\sin t}{t} = \dfrac{\pi}{4}$

51. $t = x - 1$; $\sin(\pi x) = \sin(\pi t + \pi) = -\sin \pi t$; and $\lim\limits_{x\to 1}\dfrac{\sin(\pi x)}{x-1} = -\lim\limits_{t\to 0}\dfrac{\sin \pi t}{t} = -\pi$

52. $t = x - \pi/4$; $\tan x - 1 = \dfrac{2\sin t}{\cos t - \sin t}$; $\lim\limits_{x\to\pi/4}\dfrac{\tan x - 1}{x - \pi/4} = \lim\limits_{t\to 0}\dfrac{2\sin t}{t(\cos t - \sin t)} = 2$

53. $t = x - \pi/4$, $\dfrac{\cos x - \sin x}{x - \pi/4} = -\dfrac{\sqrt{2}\sin t}{t}$; $\lim\limits_{x\to\pi/4}\dfrac{\cos x - \sin x}{x - \pi/4} = -\sqrt{2}\lim\limits_{t\to 0}\dfrac{\sin t}{t} = -\sqrt{2}$

54. $\lim_{x\to 0} h(x) = L = h(0)$ so h is continuous at $x = 0$.

Apply the Theorem to $h \circ g$ to obtain on the one hand $h(g(0)) = L$, and on the other

$$h(g(x)) = \begin{cases} \dfrac{f(g(x))}{g(x)}, x \neq 0 \\ L, \qquad x = 0 \end{cases}$$ Since $f(g(x)) = x$ and $g = f^{-1}$ this shows that $\lim_{t\to 0} \dfrac{x}{f^{-1}(x)} = L$

55. $\lim_{x\to 0} \dfrac{x}{\sin^{-1} x} = \lim_{x\to 0} \dfrac{\sin x}{x} = 1$

56. $\tan(\tan^{-1} x) = x$, so $\lim_{x\to 0} \dfrac{\tan^{-1} x}{x} = \lim_{x\to 0} \dfrac{x}{\tan x} = (\lim_{x\to 0} \cos x) \lim_{x\to 0} \dfrac{x}{\sin x} = 1$

57. $5 \lim_{x\to 0} \dfrac{\sin^{-1} x}{5x} = 5 \lim_{x\to 0} \dfrac{5x}{\sin 5x} = 5$

58. $\lim_{x\to 1} \dfrac{1}{x+1} \lim_{x\to 1} \dfrac{\sin^{-1}(x-1)}{x-1} = \dfrac{1}{2} \lim_{x\to 1} \dfrac{x-1}{\sin(x-1)} = \dfrac{1}{2}$

59. $3 \lim_{x\to 0} \dfrac{e^{3x} - 1}{3x} = 3$

60. With $y = \ln(1+x), e^{\ln(1+x)} = 1 + x, x = e^y - 1$, so $\lim_{x\to 0} \dfrac{\ln(1+x)}{x} = \lim_{x\to 0} \dfrac{x}{e^x - 1} = 1$

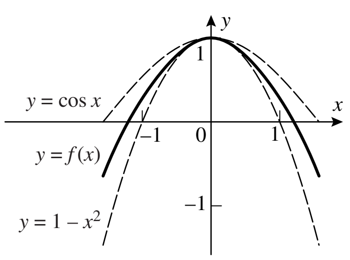
61. $(\lim_{x\to 0} x) \lim_{x\to 0} \dfrac{e^{x^2} - 1}{x^2} = 0 \cdot 1 = 0$

62. $5 \lim_{x\to 0} \dfrac{\ln(1+5x)}{5x} = 5 \cdot 1 = 5$ (Exercise 60)

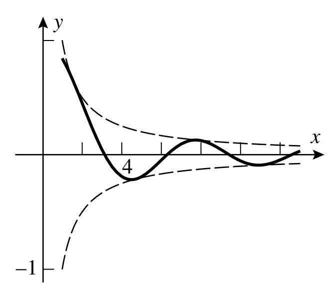
63. $-|x| \leq x \cos\left(\dfrac{50\pi}{x}\right) \leq |x|$

64. $-x^2 \leq x^2 \sin\left(\dfrac{50\pi}{\sqrt[3]{x}}\right) \leq x^2$

65. $\lim_{x\to 0} f(x) = 1$ by the Squeezing Theorem

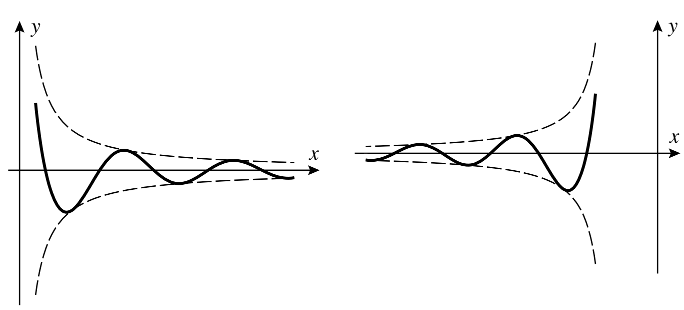


66. $\lim_{x\to +\infty} f(x) = 0$ by the Squeezing Theorem



67. Let $g(x) = -\dfrac{1}{x}$ and $h(x) = \dfrac{1}{x}$; thus $\lim_{x\to +\infty} \dfrac{\sin x}{x} = 0$ by the Squeezing Theorem.

68.



69. **(a)** $\sin x = \sin t$ where x is measured in degrees, t is measured in radians and $t = \dfrac{\pi x}{180}$. Thus $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{t\to 0} \dfrac{\sin t}{(180t/\pi)} = \dfrac{\pi}{180}$.

70. $\cos x = \cos t$ where x is measured in degrees, t in radians, and $t = \dfrac{\pi x}{180}$. Thus $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = \lim\limits_{t\to 0} \dfrac{1-\cos t}{(180t/\pi)} = 0$.

71. **(a)** $\sin 10° = 0.17365$ **(b)** $\sin 10° = \sin \dfrac{\pi}{18} \approx \dfrac{\pi}{18} = 0.17453$

72. **(a)** $\cos\theta = \cos 2\alpha = 1 - 2\sin^2(\theta/2)$
$\approx 1 - 2(\theta/2)^2 = 1 - \frac{1}{2}\theta^2$

(b) $\cos 10° = 0.98481$

(c) $\cos 10° = 1 - \dfrac{1}{2}\left(\dfrac{\pi}{18}\right)^2 \approx 0.98477$

73. **(a)** 0.08749 **(b)** $\tan 5° \approx \dfrac{\pi}{36} = 0.08727$

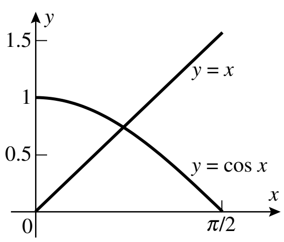
74. **(a)** $h = 52.55$ ft

(b) Since α is small, $\tan\alpha° \approx \dfrac{\pi\alpha}{180}$ is a good approximation.

(c) $h \approx 52.36$ ft

75. **(a)** Let $f(x) = x - \cos x$; $f(0) = -1$, $f(\pi/2) = \pi/2$. By the IVT there must be a solution of $f(x) = 0$.

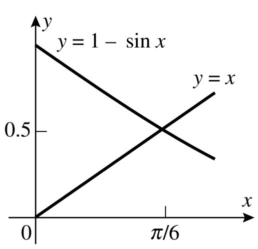
(b)



(c) 0.739

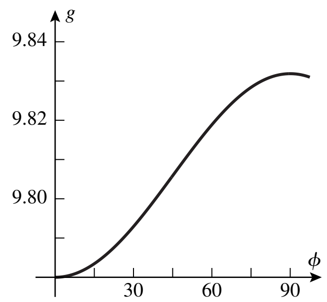
76. **(a)** $f(x) = x + \sin x - 1$; $f(0) = -1$, $f(\pi/6) = \pi/6 - 1/2 > 0$. By the IVT there must be a solution of $f(x) = 0$ in the interval.

(b)



(c) $x = 0.511$

77. **(a)** Gravity is stronger at the poles than at the equator.



(b) Let $g(\phi)$ be the given function. Then $g(38) < 9.8$ and $g(39) > 9.8$, so by the Intermediate Value Theorem there is a value c between 38 and 39 for which $g(c) = 9.8$ exactly.

78. **(a)** does not exist **(b)** the limit is zero

(c) For part (a) consider the fact that given any $\delta > 0$ there are infinitely many rational numbers x satisfying $|x| < \delta$ and there are infinitely many irrational numbers satisfying the same condition. Thus if the limit were to exist, it could not be zero because of the rational numbers, and it could not be 1 because of the irrational numbers, and it could not be anything else because of *all* the numbers. Hence the limit cannot exist. For part (b) use the Squeezing Theorem with $+x$ and $-x$ as the 'squeezers'.

REVIEW EXERCISES CHAPTER 2

1. **(a)** 1 **(b)** no limit **(c)** no limit
(d) 1 **(e)** 3 **(f)** 0
(g) 0 **(h)** 2 **(i)** 1/2

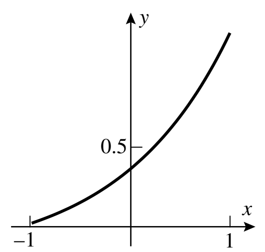
2. **(a)** $0.222\ldots, 0.24390, 0.24938, 0.24994, 0.24999, 0.25000$; for $x \neq 2$, $f(x) = \dfrac{1}{x+2}$, so the limit is 1/4.

(b) $1.15782, 4.22793, 4.00213, 4.00002, 4.00000, 4.00000$; to prove, use $\dfrac{\tan 4x}{x} = \dfrac{\sin 4x}{x \cos 4x} = \dfrac{4}{\cos 4x}\dfrac{\sin 4x}{4x}$, the limit is 4.

3. **(a)**

x	1	0.1	0.01	0.001	0.0001	0.00001	0.000001
$f(x)$	1.000	0.443	0.409	0.406	0.406	0.405	0.405

(b)



4.

x	3.1	3.01	3.001	3.0001	3.00001	3.000001
$f(x)$	5.74	5.56	5.547	5.545	5.5452	5.54518

A CAS yields 5.545177445

5. 1

6. For $x \neq 1$, $\dfrac{x^3 - x^2}{x - 1} = x^2$, so $\lim\limits_{x \to -1} \dfrac{x^3 - x^2}{x - 1} = 1$

7. If $x \neq -3$ then $\dfrac{3x + 9}{x^2 + 4x + 3} = \dfrac{3}{x + 1}$ with limit $-\dfrac{3}{2}$

8. $-\infty$

9. $\dfrac{2^5}{3} = \dfrac{32}{3}$

10. $\dfrac{\sqrt{x^2+4} - 2}{x^2} \dfrac{\sqrt{x^2+4} + 2}{\sqrt{x^2+4} + 2} = \dfrac{x^2}{x^2(\sqrt{x^2+4} + 2)} = \dfrac{1}{\sqrt{x^2+4} + 2}$, so

$$\lim_{x \to 0} \frac{\sqrt{x^2+4} - 2}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+4} + 2} = \frac{1}{4}$$

11. **(a)** $y = 0$ **(b)** none **(c)** $y = 2$

12. **(a)** $\sqrt{5}$, no limit, $\sqrt{10}$, $\sqrt{10}$, no limit, $+\infty$, no limit
(b) $-1, +1, -1, -1$, no limit, $-1, +1$

13. 1

14. 2

15. $3 - k$

16. $\lim\limits_{\theta \to 0} \tan\left(\dfrac{1 - \cos\theta}{\theta}\right) = \tan\left(\lim\limits_{\theta \to 0} \dfrac{1 - \cos\theta}{\theta}\right) = \tan\left(\lim\limits_{\theta \to 0} \dfrac{1 - \cos^2\theta}{\theta(1 + \cos\theta)}\right) = \tan 0 = 0$

17. $+\infty$

18. $\ln(2\sin\theta\cos\theta) - \ln\tan\theta = \ln 2 + 2\ln\cos\theta$ so the limit is $\ln 2$.

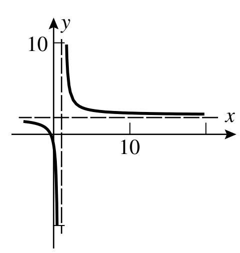
19. $\left(1 + \dfrac{3}{x}\right)^{-x} = \left[\left(1 + \dfrac{3}{x}\right)^{x/3}\right]^{(-3)}$ so the limit is e^{-3}

20. $\left(1 + \dfrac{a}{x}\right)^{bx} = \left[\left(1 + \dfrac{a}{x}\right)^{x/a}\right]^{(ab)}$ so the limit is e^{ab}

21. \$2,001.60, \$2,009.66, \$2,013.62, \$2013.75

23. **(a)** $f(x) = 2x/(x - 1)$

(b)



24. Given any window of height 2ϵ centered at the point $x = a, y = L$ there exists a width 2δ such that the window of width 2δ and height 2ϵ contains all points of the graph of the function for x in that interval.

25. **(a)** $\lim\limits_{x \to 2} f(x) = 5$ **(b)** 0.0045

26. $\delta \approx 0.07747$ (use a graphing utility)

27. **(a)** $|4x - 7 - 1| < 0.01, 4|x - 2| < 0.01, |x - 2| < 0.0025$, let $\delta = 0.0025$

(b) $\left|\dfrac{4x^2 - 9}{2x - 3} - 6\right| < 0.05, |2x + 3 - 6| < 0.05, |x - 1.5| < 0.025$, take $\delta = 0.025$

(c) $|x^2 - 16| < 0.001$; if $\delta < 1$ then $|x + 4| < 9$ if $|x - 4| < 1$; then $|x^2 - 16| = |x - 4||x + 4| \le 9|x - 4| < 0.001$ provided $|x - 4| < 0.001/9$, take $\delta = 0.0001$, then $|x^2 - 16| < 9|x - 4| < 9(0.0001) = 0.0009 < 0.001$

28. **(a)** Given $\epsilon > 0$ then $|4x - 7 - 1| < \epsilon$ provided $|x - 2| < \epsilon/4$, take $\delta = \epsilon/4$

(b) Given $\epsilon > 0$ the inequality $\left|\dfrac{4x^2 - 9}{2x - 3} - 6\right| < \epsilon$ holds if $|2x + 3 - 6| < \epsilon, |x - 1.5| < \epsilon/2$, take $\delta = \epsilon/2$

29. Let $\epsilon = f(x_0)/2 > 0$; then there corresponds $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$, $-\epsilon < f(x) - f(x_0) < \epsilon$, $f(x) > f(x_0) - \epsilon = f(x_0)/2 > 0$ for $x_0 - \delta < x < x_0 + \delta$.

30. **(a)**

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
$f(x)$	0.49	0.54	0.540	0.5403	0.54030	0.54030

(b) $\cos 1$

31. **(a)** f is not defined at $x = \pm 1$, continuous elsewhere

(b) none

(c) f is not defined at $x = 0, -3$

32. **(a)** continuous everywhere except $x = \pm 3$

(b) defined and continuous for $x \le -1$, $x \ge 1$

(c) continuous for $x > 0$

33. For $x < 2$ f is a polynomial and is continuous; for $x > 2$ f is a polynomial and is continuous. At $x = 2$, $f(2) = -13 \ne 13 = \lim_{x \to 2^+} f(x)$ so f is not continuous there.

35. $f(x) = -1$ for $a \le x < \dfrac{a + b}{2}$ and $f(x) = 1$ for $\dfrac{a + b}{2} \le x \le b$

36. If, on the contrary, $f(x_0) < 0$ for some x_0 in $[0, 1]$, then by the Intermediate Value Theorem we would have a solution of $f(x) = 0$ in $[0, x_0]$, contrary to the hypothesis.

37. $f(-6) = 185$, $f(0) = -1$, $f(2) = 65$; apply Theorem 2.4.8 twice, once on $[-6, 0]$ and once on $[0, 2]$