Chapter 2

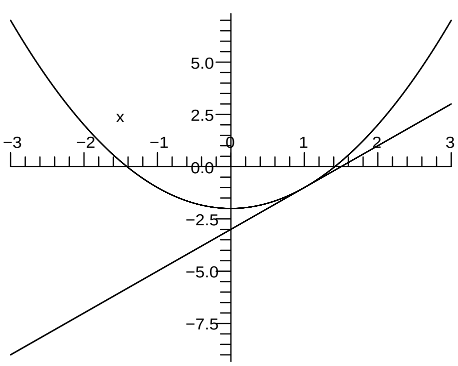
Differentiation

2.1 Tangent Lines and Velocity

1. Slope is

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{(1+h)^2-2-(-1)}{h}$$
$$= \lim_{h\to 0} \frac{h^2+2h}{h}$$
$$= \lim_{h\to 0} (h+2) = 2.$$

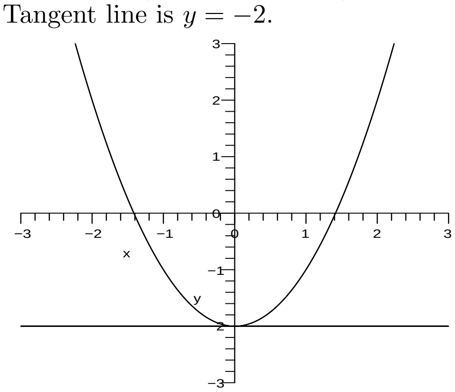
Tangent line is $y = 2(x-1)-1$ or $y = 2x-3$.



2. Slope is

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2}{h} = 0.$$

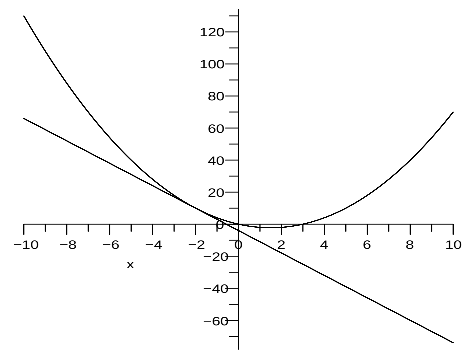
Tangent line is $y = -2$.



3. Slope is

$$\lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h}$$
$$= \lim_{h\to 0} \frac{(-2+h)^2-3(-2+h)-(10)}{h}$$
$$= \lim_{h\to 0} \frac{-7h+h^2}{h} = -7.$$

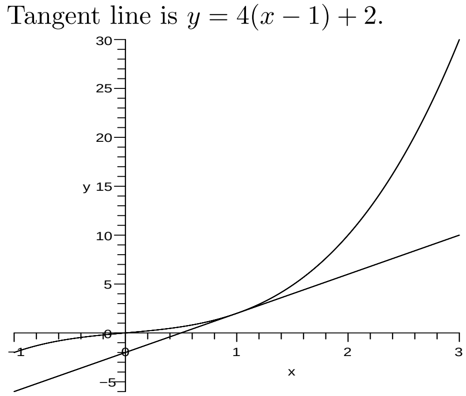
Tangent line is $y = -7(x+2)+10$



4. Slope is

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{(1+3h+3h^2+h^3)+(1+h)-2}{h}$$
$$= \lim_{h\to 0} \frac{4h+3h^2+h^3}{h} = \lim_{h\to 0} 4+3h+h^2 = 4.$$

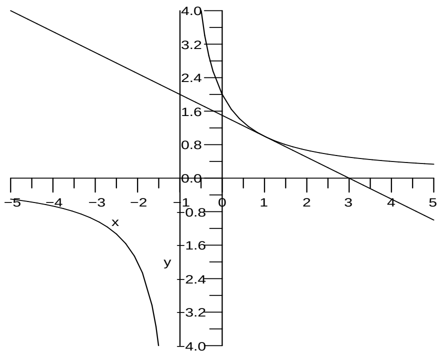
Tangent line is $y = 4(x-1)+2$.



5. Slope is

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2}{(1+h)+1}-\frac{2}{1+1}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2}{2+h}-1}{h} = \lim_{h\to 0} \frac{\left(\frac{2-(2+h)}{2+h}\right)}{h}$$
$$= \lim_{h\to 0} \frac{\left(\frac{-h}{2+h}\right)}{h} = \lim_{h\to 0} \frac{-1}{2+h} = -\frac{1}{2}.$$

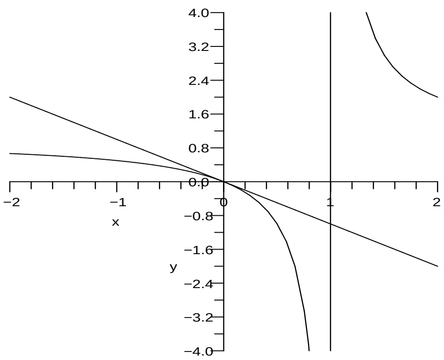
Tangent line is $y = -\frac{1}{2}(x-1)+1$ or $y = -\frac{x}{2}+\frac{3}{2}$.



6. Slope is

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{h}{h-1}-0}{h}$$
$$=\lim_{h\to 0}\frac{1}{h-1}=-1$$

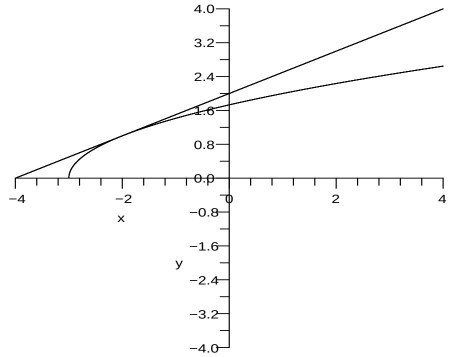
Tangent line is $y=-x$.



7. Slope is

$$\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{(-2+h)+3}-1}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}\cdot\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}$$
$$=\lim_{h\to 0}\frac{(h+1)-1}{h(\sqrt{h+1}+1)}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{h+1}+1}=\frac{1}{2}.$$

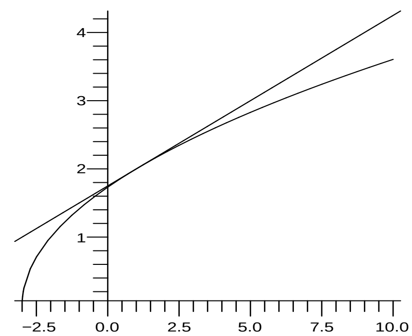
Tangent line is $y=\frac{1}{2}(x+2)+1$ or $y=\frac{1}{2}x+2$.



8. Slope is

$$\lim_{h\to 0}\frac{f(1+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{(1+h)+3}-\sqrt{1+3}}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+4}-2}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+4}-2}{h}\cdot\frac{\sqrt{h+4}+2}{\sqrt{h+4}+2}$$
$$=\lim_{h\to 0}\frac{h+4-4}{h}\cdot\frac{1}{\sqrt{h+4}+2}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{h+4}+2}=\frac{1}{4}.$$

Tangent line is $y=\frac{1}{4}(x-1)+2$.



9. $f(x)=x^3-x$

No.	Points $(x,\ y)$	Slope
(a)	(1,0) and (2,6)	6
(b)	(2,6) and (3,24)	18
(c)	(1.5,1.875) and (2,6)	8.25
(d)	(2,6) and (2.5,13.125)	14.25
(e)	(1.9,4.959) and (2,6)	10.41
(f)	(2,6) and (2.1,7.161)	11.61

(g) Slope seems to be approximately 11.

10. $f(x)=\sqrt{x^2+1}$

No.	Points (x, y)	Slope
(a)	(1,1.414) and (2,2.236)	0.504
(b)	(2,2.236) and (3,3.162)	0.926
(c)	(1.5,1.803) and (2,2.236)	0.867
(d)	(2,2.236) and (2.5,2.269)	0.913
(e)	(1.9,2.147) and (2,2.236)	0.89
(f)	(2,2.236) and (2.1,2.325)	0.899

(g) Slope seems to be approximately 0.89.

11. $f(x) = \dfrac{x-1}{x+1}$

No.	Points (x, y)	Slope
(a)	(1,0) and (2,0.33)	0.33
(b)	(2,0.33) and (3,0.5)	0.17
(c)	(1.5,0.2) and (2,0.33)	0.26
(d)	(2,0.33) and (2.5,0.43)	0.2
(e)	(1.9,0.31) and (2,0.33)	0.2
(f)	(2,0.33) and (2.1,0.35)	0.2

(g) Slope seems to be approximately 0.2.

12. $f(x) = \dfrac{2}{x}$

No.	Points (x, y)	Slope
(a)	(1, 2) and (2,1)	−1
(b)	(2,1) and (3,0.6667)	−0.3333
(c)	(1.5,1.3333) and (2,1)	−0.6667
(d)	(2,1) and (2.5,0.8)	−0.4
(e)	(1.9,1.0526) and (2,1)	−0.5263
(f)	(2,1) and (2.1,0.9524)	−0.4762

(g) Slopes seems to be approximately 0.5013

13. C, B, A, D. At the point labeled C, the slope is very steep and negative. At the point B, the slope is zero and at the point A, the slope is just more than zero. The slope of the line tangent to the point D is large and positive.

14. In order of increasing slope: D (large negative), C (small negative), B (small positive), and A (large positive).

15. (a) Velocity at time $t = 1$ is,

$$\lim_{h\to 0} \frac{s(1+h)-s(1)}{h}$$
$$= \lim_{h\to 0} \frac{-4.9(1+h)^2+5-(0.1)}{h}$$
$$= \lim_{h\to 0} \frac{-4.9(1+2h+h^2)+5-(0.1)}{h}$$
$$= \lim_{h\to 0} \frac{-9.8h-4.9h^2}{h}$$
$$= \lim_{h\to 0} \frac{h\,(-9.8-4.9h)}{h} = -9.8.$$

(b) Velocity at time $t = 2$ is,

$$\lim_{h\to 0} \frac{s(2+h)-s(2)}{h}$$
$$= \lim_{h\to 0} \frac{-4.9(2+h)^2+5-(-14.6)}{h}$$
$$= \lim_{h\to 0} \frac{-4.9(4+4h+h^2)+5-(-14.6)}{h}$$
$$= \lim_{h\to 0} \frac{-19.6h-4.9h^2}{h}$$
$$= \lim_{h\to 0} \frac{h\,(-19.6-4.9h)}{h} = -19.6$$

16. (a) Velocity at time $t = 0$ is,

$$\lim_{h\to 0} \frac{s(0+h)-s(0)}{h}$$
$$= \lim_{h\to 0} \frac{4h-4.9h^2}{h}$$
$$= \lim_{h\to 0} \frac{h\,(4-4.9h)}{h}$$
$$= 4 - \lim_{h\to 0} 4.9h = 4.$$

(b) Velocity at time $t = 1$ is,

$$\lim_{h\to 0} \frac{s(1+h)-s(1)}{h}$$
$$= \lim_{h\to 0} \frac{4(1+h)-4.9(1+h)^2-(-0.9)}{h}$$
$$= \lim_{h\to 0} \frac{4+4h-4.9-9.8h-4.9h^2+0.9}{h}$$
$$= \lim_{h\to 0} \frac{-5.8h-4.9h^2}{h}$$
$$= \lim_{h\to 0} \frac{h\,(-5.8-4.9h)}{h} = -5.8$$

17. (a) Velocity at time $t = 0$ is,

$$\lim_{h\to 0} \frac{s(0+h)-s(0)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{h+16}-4}{h} \cdot \frac{\sqrt{h+16}+4}{\sqrt{h+16}+4}$$
$$= \lim_{h\to 0} \frac{(h+16)-16}{h(\sqrt{h+16}+4)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{h+16}+4} = \frac{1}{8}$$

(b) Velocity at time $t = 2$ is,

$$\lim_{h\to 0} \frac{s(2+h)-s(2)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{18+h}-\sqrt{18}}{h}$$

Multiplying by $\dfrac{\sqrt{h+18}+\sqrt{18}}{\sqrt{h+18}+\sqrt{18}}$ gives

$$= \lim_{h\to 0} \frac{(h+18)-18}{h(\sqrt{h+18}+\sqrt{18})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{h+18}+\sqrt{18}} = \frac{1}{2\sqrt{18}}$$

18. (a) Velocity at time $t = 2$ is,

$$\lim_{h\to 0} \frac{s(2+h)-s(2)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4}{(2+h)} - 2}{h} = \lim_{h\to 0} \frac{\frac{4-4-2h}{(2+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-2h}{h(2+h)} = \lim_{h\to 0} \frac{-2}{2+h} = -1.$$

(b) Velocity at time $t = 4$ is,
$$\lim_{h\to 0} \frac{s(4+h) - s(4)}{h} = \lim_{h\to 0} \frac{\frac{4}{(4+h)} - 1}{h}$$
$$= \lim_{h\to 0} \frac{\frac{4-1(4+h)}{(4+h)}}{h} = \lim_{h\to 0} \frac{\frac{4-4-h}{(4+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-h}{h(4+h)} = \lim_{h\to 0} \frac{-1}{4+h} = -\frac{1}{4}$$

19. **(a)** Points: (0, 10) and (2, 74)
Average velocity: $\dfrac{74-10}{2} = 32$

(b) Second point: (1, 26)
Average velocity: $\dfrac{74-26}{1} = 48$

(c) Second point: (1.9, 67.76)
Average velocity: $\dfrac{74-67.76}{0.1} = 62.4$

(d) Second point: (1.99, 73.3616)
Average velocity: $\dfrac{74-73.3616}{0.01} = 63.84$

(e) The instantaneous velocity seems to be 64.

20. **(a)** Points: (0, 0) and (2, 26)
Average velocity: $\dfrac{26-0}{2-0} = 13$

(b) Second point: (1, 4)
Average velocity: $\dfrac{26-4}{2-1} = 22$

(c) Second point: (1.9, 22.477)
Average velocity: $\dfrac{26-22.477}{2-1.9} = 35.23$

(d) Second point: (1.99, 25.6318)
Average velocity:
$\dfrac{26-25.6318}{2-1.99} = 36.8203$

(e) The instantaneous velocity seems to be approaching 37.

21. **(a)** Points: (0, 0) and $(2, \sqrt{20})$
Average velocity: $\dfrac{\sqrt{20}-0}{2-0} = 2.236068$

(b) Second point: (1, 3)
Average velocity: $\dfrac{\sqrt{20}-3}{2-1} = 1.472136$

(c) Second point: $(1.9, \sqrt{18.81})$
Average velocity:
$\dfrac{\sqrt{20}-\sqrt{18.81}}{2-1.9} = 1.3508627$

(d) Second point: $(1.99, \sqrt{19.8801})$
Average velocity:
$\dfrac{\sqrt{20}-\sqrt{19.88}}{2-1.99} = 1.3425375$

(e) One might conjecture that these numbers are approaching 1.34. The exact limit is $\dfrac{6}{\sqrt{20}} \approx 1.341641$.

22. **(a)** Points: (0, −2.7279) and (2, 0)
Average velocity:
$\dfrac{0-(-2.7279)}{2-0} = 1.3639$

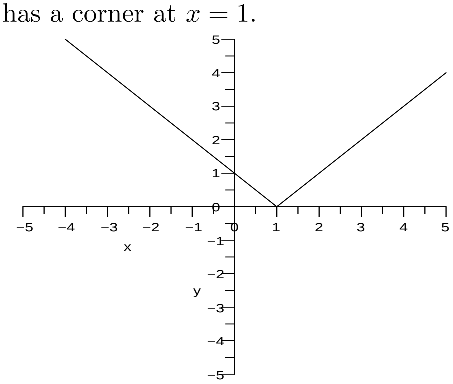
(b) Second point: (1, −2.5244)
Average velocity:
$\dfrac{0-(-2.5244)}{2-1} = 2.5244$

(c) Second point: (1.9, −0.2995)
Average velocity:
$\dfrac{0-(-0.2995)}{2-1.9} = 2.995$

(d) Second point: (1.99, −0.03)
Average velocity: $\dfrac{0-(-0.03)}{2-1.99} = 3$

(e) The instantaneous velocity seems to be 3.

23. A graph makes it apparent that this function has a corner at $x = 1$.



Numerical evidence suggests that,
$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = 1$$