PROJECTS

Project 1 Harvesting a Renewable Resource

Suppose that the population y of a certain species of fish (for example, tuna or halibut) in a given area of the ocean is described by the logistic equation

$$dy/dt = r(1 - y/K)y.$$

If the population is subjected to harvesting at a rate $H(y, t)$ members per unit time, then the harvested population is modeled by the differential equation

$$dy/dt = r(1 - y/K)y - H(y, t). \tag{1}$$

Although it is desirable to utilize the fish as a food source, it is intuitively clear that if too many fish are caught, then the fish population may be reduced below a useful level and possibly even driven to extinction. The following problems explore some of the questions involved in formulating a rational strategy for managing the fishery.

Project 1 PROBLEMS

1. **Constant Effort Harvesting**. At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population y: the more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by $H(y, t) = Ey$, where E is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. With this choice for $H(y, t)$, Eq. (1) becomes

$$dy/dt = r(1 - y/K)y - Ey. \qquad \text{(i)}$$

This equation is known as the **Schaefer model** after the biologist M. B. Schaefer, who applied it to fish populations.

(a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K(1-E/r) > 0$.

(b) Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population y_2. Find Y as a function of the effort E. The graph of this function is known as the yield–effort curve.

(d) Determine E so as to maximize Y and thereby find the **maximum sustainable yield** Y_m.

2. **Constant Yield Harvesting**. In this problem, we assume that fish are caught at a constant rate h independent of the size of the fish population, that is, the harvesting rate $H(y, t) = h$. Then y satisfies

$$dy/dt = r(1 - y/K)y - h = f(y). \qquad \text{(ii)}$$

The assumption of a constant catch rate h may be reasonable when y is large but becomes less so when y is small.

(a) If $h < rK/4$, show that Eq. (ii) has two equilibrium points y_1 and y_2 with $y_1 < y_2$; determine these points.

(b) Show that y_1 is unstable and y_2 is asymptotically stable.

(c) From a plot of $f(y)$ versus y, show that if the initial population $y_0 > y_1$, then $y \to y_2$ as $t \to \infty$, but if $y_0 < y_1$, then y decreases as t increases. Note that $y = 0$ is not an equilibrium point, so if $y_0 < y_1$, then extinction will be reached in a finite time.

(d) If $h > rK/4$, show that y decreases to zero as t increases regardless of the value of y_0.

(e) If $h = rK/4$, show that there is a single equilibrium point $y = K/2$ and that this point is semistable (see Problem 7, Section 2.5). Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $y = K/2$. Observe that h_m has the same value as Y_m in Problem 1(d). The fishery is considered to be overexploited if y is reduced to a level below $K/2$.

Project 2 Designing a Drip Dispenser for a Hydrology Experiment

In order to make laboratory measurements of water filtration and saturation rates in various types of soils under the condition of steady rainfall, a hydrologist wishes to design drip dispensing containers in such a way that the water drips out at a nearly constant rate. The containers are supported above glass cylinders that contain the soil samples (Figure 2.P.1). The hydrologist elects to use the following differential equation, based on Torricelli's principle (see Problem 6, Section 2.3), to help solve the design problem,

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}. \qquad (1)$$

In Eq. (1), $h(t)$ is the height of the liquid surface above the dispenser outlet at time t, $A(h)$ is the cross-sectional area of the dispenser at height h, a is the area of the outlet, and α is a measured contraction coefficient that accounts for the observed fact that the cross section of the (smooth) outflow stream is smaller than a. Note that the hydrologist is using a laminar flow model as a guide in designing the shape of the container. Forces due to surface tension at the tiny outlet are ignored in the design problem. Once the shape

FIGURE 2.P.1 Water dripping into a soil sample.

of the container has been determined, the outlet aperture is adjusted to a desired drip rate that will remain nearly constant for an extended period of time. Of course, since surface tension forces are not accounted for in Eq. (1), the equation is not a valid model for the output flow rate when the aperture is so small that the water drips out. Nevertheless, once the hydrologist sees and interprets the results based on her design strategy, she feels justified in using Eq. (1).

Project 2 PROBLEMS

1. Assume that the shape of the dispensers are surfaces of revolution so that $A(h) = \pi[r(h)]^2$ where $r(h)$ is the radius of the container at height h. For each of the h dependent cross-sectional radii prescribed below in (i)–(v),

CAS **(a)** create a surface plot of the surface of revolution, and

ODEA **(b)** find numerical approximations of solutions of Eq. (1) for $0 \leq t \leq 60$:

i. $r(h) = r_1, \quad 0 \leq h \leq H$

ii. $r(h) = r_0 + (r_1 - r_0)h/H, \quad 0 \leq h \leq H$

iii. $r(h) = r_0 + (r_1 - r_0)\sqrt{h/H}$, $0 \leq h \leq H$

iv. $r(h) = r_0 \exp[(h/H)\ln(r_1/r_0)]$ $0 \leq h \leq H$

v. $r(h) = \dfrac{r_0 r_1 H}{r_1 H - (r_1 - r_0)h}, \quad 0 \leq h \leq H.$

Use the parameter values specified in the following table:

$$r_0 = 0.1 \text{ ft}$$
$$r_1 = 1 \text{ ft}$$
$$\alpha = 0.6$$
$$a = 0.1 \text{ ft}^2$$

In addition, use the initial condition $h(0) = H$, where the initial height H of water in each of the containers is determined by requiring that the initial volume of water satisfies

$$V(0) = \int_0^H \pi r^2(h)\,dh = 1 \text{ ft}^3.$$

Determine the qualitative shape of the container such that the output flow rate given by the right-hand side of Eq. (1), $F_R(t) = \alpha a\sqrt{2gh(t)}$, varies slowly during the early stages of the experiment. As a design criterion, consider plotting the ratio $R = F_R(t)/F_R(0)$ for $0 \leq t \leq 60$, where values of R near 1 are most desirable. Based on the results of your computer experiments, sketch the shape of what a suitable container should look like.

2. After viewing the results of her computer experiments, it slowly dawns on the hydrologist that the "optimal shape" of the container is consistent with what would be expected based on the conceptualization that the water in the ideal container would consist of a collection of small parcels of water of mass m, all possessing the same amount of potential energy. If laboratory spatial constraints were not ODEA

an issue, what would be the ideal "shape" of each container? Perform a computer experiment that supports your conclusions based on potential energy considerations.

Project 3 A Mathematical Model of a Groundwater Contaminant Source

Chlorinated solvents such as trichloroethylene (TCE) are a common cause of environmental contamination[12] at thousands of government and private industry facilities. TCE and other chlorinated organics, collectively referred to as dense nonaqueous phase liquids (DNAPLs), are denser than water and only slightly soluble in water. DNAPLs tend to accumulate as a separate phase below the water table and provide a long-term source of groundwater contamination. A downstream contaminant plume is formed by the process of dissolution of DNAPL into water flowing through the source region as shown in Figure 2.P.2.

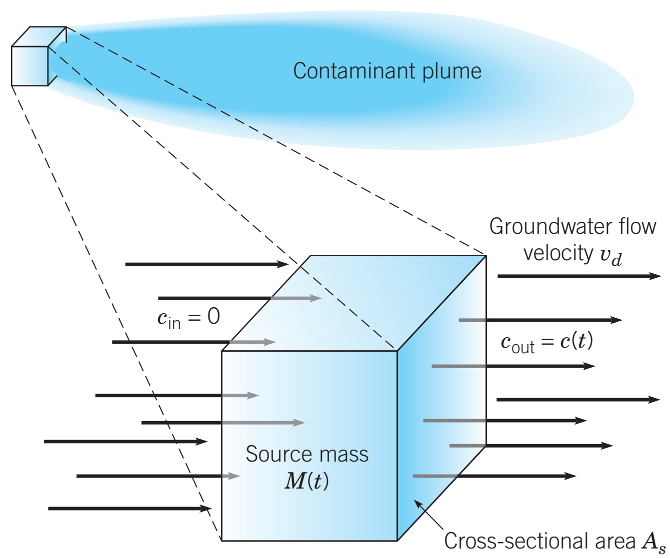

FIGURE 2.P.2 Conceptual model of DNAPL source.

In this project, we study a first order differential equation that describes the time dependent rate of dissolved contaminant discharge leaving the source zone and entering the plume.[13]

Parameters and variables relevant to formulating a mathematical model of contaminant discharge from the source region are defined in the following table:

$$A_s = \text{cross-sectional area of the source region}$$
$$v_d = \text{Darcy groundwater flow velocity.}^{14}$$
$$m(t) = \text{total DNAPL mass in source region}$$

[12] Falta, R.W., P.S. Rao, and N. Basu, "Assessing the Impacts of Partial Mass Depletion in DNAPL Source Zones: I. Analytical Modeling of Source Strength Functions and Plume Response," *Journal of Contaminant Hydrology* 78, (2005)(4) pp 259–280.

[13] The output of this model can then be used as input into another mathematical model that in turn describes the processes of advection, adsorption, dispersion, and degradation of contaminant within the plume.

[14] In porous media flow, the Darcy flow velocity v_d is defined by $v_d = Q/A$ where A is a cross-sectional area available for flow and Q is the volumetric flow rate (volume/time) through A.

$c_s(t)$ = concentration (flow averaged) of dissolved contaminant leaving the source zone

m_0 = initial DNAPL mass in source region

c_0 = source zone concentration (flow averaged) corresponding to an initial source zone mass of m_0

The equation describing the rate of DNAPL mass discharge from the source region is

$$\frac{dm}{dt} = -A_s v_d c_s(t), \tag{1}$$

while an algebraic relationship between $c_s(t)$ and $m(t)$ is postulated in the form of a power law,

$$\frac{c_s(t)}{c_0} = \left[\frac{m(t)}{m_0}\right]^{\gamma}, \tag{2}$$

in which $\gamma > 0$ is empirically determined. Combining Eqs. (1) and (2) (Problem 1) yields a first order differential equation

$$\frac{dm}{dt} = -\alpha m^{\gamma} \tag{3}$$

that models the dissolution of DNAPL into the groundwater flowing through the source region.

Project 3 PROBLEMS

1. Derive Eq. (3) from Eqs. (1) and (2) and show that $\alpha = v_d A_s c_0 / m_0^{\gamma}$.
2. Additional processes due to biotic and abiotic degradation contributing to source decay can be accounted for by adding a decay term to (3) that is proportional to $m(t)$,

$$m'(t) = -\alpha m^{\gamma} - \lambda m, \tag{i}$$

where λ is the associated decay rate constant. Find solutions of Eq. (i) using the initial condition $m(0) = m_0$ for the following cases: (i) $\gamma = 1$, (ii) $\gamma \neq 1$ and $\lambda = 0$, (iii) $\gamma \neq 1$ and $\lambda \neq 0$. Then find expressions for $c_s(t)$ using Eq. (2).
Hint: Eq. (i) is a type of nonlinear equation known as a Bernoulli equation. A method for solving Bernoulli equations is discussed in Problem 27 of Section 2.4.
3. Show that when $\gamma \geq 1$ the source has an infinite lifetime but if $0 < \gamma < 1$ the source has a finite lifetime. In the latter case, find the time that the DNAPL source mass attains the value zero.
4. ODEA Assume the following values for the parameters: $m_0 = 1620$ kg, $c_0 = 100$ mg/l, $A_s = 30$ m^2, $v_d = 20$ m/yr, $\lambda = 0$. Use the solutions obtained in Problem 2 to plot graphs of $c_s(t)$ for each of the following cases: (i) $\gamma = 0.5$ for $0 \leq t \leq t_f$ where $c_s(t_f) = 0$, (ii) $\gamma = 2$ for $0 \leq t \leq 100$ years.
5. **Effects of Partial Source Remediation.** ODEA
   (a) Assume that a source remediation process results in a 90% reduction in the initial amount of DNAPL mass in the source region. Repeat Problem 4 with m_0 and c_0 in Eq. (2) replaced by $m_1 = (0.1)\, m_0$ and $c_1 = (0.1)^{\gamma} c_0$, respectively. Compare the graphs of $c_s(t)$ in this case with the graphs obtained in Problem 4.
   (b) Assume that the 90% efficient source remediation process is not applied until $t_1 = 10$ years have elapsed following the initial deposition of the contaminant. Under this scenario, plot the graphs of $c_s(t)$ using the parameters and initial conditions of Problem 4. In this case, use Eq. (2) to compute concentration for $0 \leq t < t_1$. Following remediation, use the initial condition $m(t_1) = m_1 = 0.1 m(t_1 - 0) = 0.1 \lim_{t \uparrow t_1} m(t)$ for Eq. (i) and use the following modification of Eq. (2),

$$\frac{c_s(t)}{c_1} = \left[\frac{m(t)}{m_1}\right]^{\gamma}, \qquad t > t_1, \tag{5}$$

   where $c_1 = (0.1)^{\gamma} c(t_1 - 0) = (0.1)^{\gamma} \lim_{t \uparrow t_1} c(t)$ to compute concentrations for times $t > t_1$. Compare the graphs of $c_s(t)$ in this case with the graphs obtained in Problems 4 and 5(a). Can you draw any conclusions about the possible effectiveness of source remediation? If so, what are they?

Project 4 Monte Carlo Option Pricing: Pricing Financial Options by Flipping a Coin

A discrete model for change in price of a stock over a time interval $[0, T]$ is

$$S_{n+1} = S_n + \mu\, S_n\, \Delta t + \sigma\, S_n\, \varepsilon_{n+1} \sqrt{\Delta t}, \qquad S_0 = s \tag{1}$$

where $S_n = S(t_n)$ is the stock price at time $t_n = n\Delta t,\ n = 0, \ldots, N-1$, $\Delta t = T/N$, μ is the annual growth rate of the stock, and σ is a measure of the stock's annual price volatility or tendency to fluctuate. Highly volatile stocks have large values for σ, for example, values ranging from 0.2 to 0.4. Each term in the sequence $\varepsilon_1, \varepsilon_2, \ldots$ takes on the value 1 or -1 depending on whether the outcome of a coin tossing experiment is heads or tails, respectively. Thus, for each $n = 1,2, \ldots$

$$\varepsilon_n = \begin{cases} 1 & \text{with probability} = 1/2 \\ -1 & \text{with probability} = 1/2. \end{cases} \tag{2}$$

A sequence of such numbers can easily be created by using one of the random number generators available in most mathematical computer software applications. Given such a sequence, the difference equation (1) can then be used to simulate a **sample path** or **trajectory** of stock prices, $\{s, S_1, S_2, \ldots S_N\}$. The "random" terms $\sigma S_n \varepsilon_{n+1}\sqrt{\Delta t}$ on the right-hand side of (1) can be thought of as "shocks" or "disturbances" that model fluctuations in the stock price. By repeatedly simulating stock price trajectories and computing appropriate averages, it is possible to obtain estimates of the price of a **European call option**, a type of financial derivative. A statistical simulation algorithm of this type is called a **Monte Carlo method**.

A European call option is a contract between two parties, a holder and a writer, whereby, for a premium paid to the writer, the holder acquires the right (but not the obligation) to purchase the stock at a future date T (the **expiration date**) at a price K (the **strike price**) agreed upon in the contract. If the buyer elects to exercise the option on the expiration date, the writer is obligated to sell the underlying stock to the buyer at the price K. Thus, the option has, associated with it, a **payoff function**

$$f(S) = \max(S - K, 0) \tag{3}$$

where $S = S(T)$ is the price of the underlying stock at the time T when the option expires (see Figure 2.P.3).

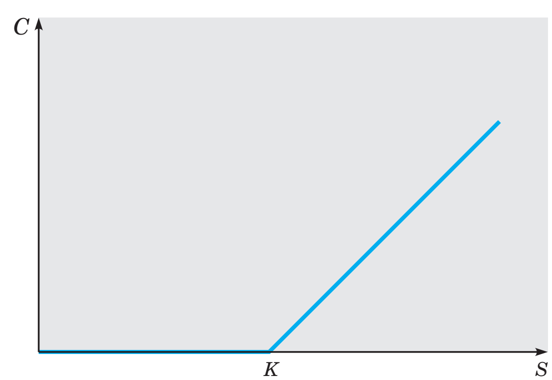

FIGURE 2.P.3 The value of a call option at expiration is $C = \max(S - K, 0)$ where K is the strike price of the option and $S = S(T)$ is the stock price at expiration.

Eq. (3) is the value of the option at time T since, if $S(T) > K$, the holder can purchase, at price K, stock with market value $S(T)$ and thereby make a profit equal to $S(T) - K$ not counting the option premium. If $S(T) < K$, the holder will simply let the option expire since it would be irrational to purchase stock at a price that exceeds the market value. The option valuation problem is to determine the correct and fair price of the option at the time that the holder and writer enter into the contract.[15]

To estimate the price of a call option using a Monte Carlo method, an ensemble

$$\left\{S_N^{(k)} = S^{(k)}(T),\ k = 1, \ldots, M\right\}$$

of M stock prices at expiration is generated using the difference equation

$$S_{n+1}^{(k)} = S_n^{(k)} + r\, S_n^{(k)}\, \Delta t + \sigma\, S_n^{(k)}\, \varepsilon_{n+1}^{(k)} \sqrt{\Delta t}, \qquad S_0^{(k)} = s. \tag{4}$$

For each $k = 1, \ldots, M$, the difference equation (4) is identical to (1) except that the growth rate μ is replaced by the annual rate of interest r that it costs the writer to borrow money. Option pricing theory requires that the average value of the payoffs $\left\{f(S_N^{(k)}),\ k = 1, \ldots, M\right\}$ be equal to the compounded total return obtained by investing the option premium, $\hat{C}(s)$, at rate r over the life of the option,

$$\frac{1}{M}\sum_{k=1}^{M} f(S_N^{(k)}) = (1 + r\,\Delta t)^N\ \hat{C}(s). \tag{5}$$

Solving (5) for $\hat{C}(s)$ yields the Monte Carlo estimate

$$\hat{C}(s) = (1 + r\,\Delta t)^{-N}\left\{\frac{1}{M}\sum_{k=1}^{M} f(S_N^{(k)})\right\} \tag{6}$$

for the option price. Thus, the Monte Carlo estimate $\hat{C}(s)$ is the present value of the average of the payoffs computed using the rules of compound interest.

Project 4 PROBLEMS

1. Show that Euler's method applied to the differential equation
$$\frac{dS}{dt} = \mu S \tag{i}$$
yields Eq. (1) in the absence of random disturbances, that is, when $\sigma = 0$.

CAS 2. Simulate five sample trajectories of (1) for the following parameter values and plot the trajectories on the same set of coordinate axes: $\mu = 0.12$, $\sigma = 0.1$, $T = 1$, $s = \$40$, $N = 254$. Then repeat the experiment using the value $\sigma = 0.25$ for the volatility. Do the sample trajectories generated in the latter case appear to exhibit a greater degree of variability in their behavior?
Hint: For the ε_n's it is permissible to use a random number generator that creates normally distributed random numbers with mean zero and variance one.

3. Use the difference equation (i) to generate an ensemble of stock prices $S_N^{(k)} = S^{(k)}(N\Delta t)$, $k = 1, \ldots, M$ (where $T = N\Delta\, t$) and then use formula (6) to compute a Monte Carlo estimate of the value of a five month call option ($T = 5/12$ years) for the following parameter values: $r = 0.06$, $\sigma = 0.2$, and $K = \$50$. Find estimates corresponding to current stock prices of $S(0) = s = \$45$, \$50, and \$55. Use $N = 200$ time steps for each trajectory and $M \cong 10{,}000$ sample trajectories for each Monte Carlo estimate.[16] Check the accuracy of your results by comparing the Monte Carlo approximation with CAS

[15]The 1997 Nobel Prize in Economics was awarded to Robert C. Merton and Myron S. Scholes for their work, along with Fischer Black, in developing the Black-Scholes options pricing model.

[16]As a rule of thumb, you may assume that the sampling error in these Monte Carlo estimates is proportional to $1/\sqrt{M}$. Using software packages such as MATLAB that allow vector operations where all M trajectories can be simulated simultaneously greatly speeds up the calculations.

the value computed from the exact Black-Scholes formula

$$C(s) = \frac{s}{2}\,\text{erfc}(-d_1/\sqrt{2}) - \frac{K}{2}\,e^{-rT}\text{erfc}(-d_2/\sqrt{2}) \qquad \text{(ii)}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln(s/k) + (r + \sigma^2/2)T\right],$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

and $\text{erfc}(x)$ is the complementary error function,

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt.$$

CAS **4. Variance Reduction by Antithetic Variates.** A simple and widely used technique for increasing the efficiency and accuracy of Monte Carlo simulations in certain situations with little additional increase in computational complexity is the method of antithetic variates. For each $k = 1, \ldots, M$ use the sequence $\left\{\varepsilon_1^{(k)}, \ldots, \varepsilon_{N-1}^{(k)}\right\}$ in (4) to simulate a payoff $f(S_N^{(k+)})$ and also use the sequence $\left\{-\varepsilon_1^{(k)}, \ldots, -\varepsilon_{N-1}^{(k)}\right\}$ in (4) to simulate an associated payoff $f(S_N^{(k-)})$. Thus, the payoffs are simulated in pairs $\left\{(f(S_N^{(k+)}), f(S_N^{(k-)}))\right\}$. A modified Monte Carlo estimate is then computed by replacing each payoff $f(S_N^{(k)})$ in (6) by the average $[f(S_N^{(k+)}) + f(S_N^{(k-)})]/2$,

$$\hat{C}_{AV}(s) = (1 + r\Delta t)^{-N}\left\{\frac{1}{M}\sum_{k=1}^{M}\frac{f(S_N^{(k+)}) + f(S_N^{(k-)})}{2}\right\}. \qquad \text{(iii)}$$

Use the parameters specified in Problem 3 to compute several (say 20 or so) option price estimates using (6) and an equivalent number of option price estimates using (iii). For each of the two methods, plot a histogram of the estimates and compute the mean and standard deviation of the estimates. Comment on the accuracies of the two methods.