Chapter 2
First Order Differential Equations

Section 2.1

1. This equation is linear because it can be written in the form $y' + p(t)y = g(t)$. It is nonhomogeneous because when it is put in this form, $g(t) \neq 0$.
2. nonlinear
3. This equation is nonlinear because it cannot be written in the form $y' + p(t)y = g(t)$.
4. nonlinear
5. This equation is nonlinear because it cannot be written in the form $y' + p(t)y = g(t)$.
6. linear, homogeneous
7. This equation is nonlinear because it cannot be written in the form $y' + p(t)y = g(t)$.
8. nonlinear
9. This equation is linear because it can be written in the form $y' + p(t)y = g(t)$. It is nonhomogeneous because when it is put in this form, $g(t) \neq 0$.
10. linear, homogeneous

11 (a). Theorem 2.1 guarantees a unique solution for the interval $(-\infty,\infty)$, since $\frac{t}{t^2+1}$ and $\sin(t)$ are both continuous for all t and -2 is on this interval.

11 (b). Theorem 2.1 guarantees a unique solution for the interval $(-\infty,\infty)$, since $\frac{t}{t^2+1}$ and $\sin(t)$ are both continuous for all t and 0 is on this interval.

11 (c). Theorem 2.1 guarantees a unique solution for the interval $(-\infty,\infty)$, since $\frac{t}{t^2+1}$ and $\sin(t)$ are both continuous for all t and π is on this interval.

12 (a). $2 < t < \infty$

12 (b). $-2 < t < 2$

12 (c). $-2 < t < 2$

12 (d). $-\infty < t < -2$

13 (a). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(3,\infty)$, the largest interval that includes $t = 5$.

13 (b). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-2,2)$, the largest interval that includes $t = -\frac{3}{2}$.

13 (c). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-2,2)$, the largest interval that includes $t = 0$.

13 (d). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-\infty,-2)$, the largest interval that includes $t = -5$.

13 (e). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-2,2)$, the largest interval that includes $t = \frac{3}{2}$.

14. $\dfrac{\ln\left|t + t^{-1}\right|}{t-2} = \dfrac{\ln\left|\frac{t^2+1}{t}\right|}{t-2}$ undefined at $t = 0,2$.

14 (a). $2 < t < \infty$.

14 (b). $0 < t < 2$.

14 (c). $-\infty < t < 0$.

14 (d). $-\infty < t < 0$.

15. $y(t) = 3e^{t^2}$. Differentiating gives us $y' = 3e^{t^2}(2t) = 2ty$. Substituting these values into the given equation yields $2ty + p(t)y = 0$. Solving this for $p(t)$, we find that $p(t) = -2t$. Putting $t = 0$ into the equation for y gives us $y_0 = 3$.

16(a). $y = Ct^r \; y' = Crt^{r-1} \qquad 2ty' - 6y = 0$

$\therefore \; 2Crt^r - 6Ct^r = (2r-6)Ct^r = 0 \;\Rightarrow\; (2r-6)y = 0 \;\Rightarrow\; 2r - 6 = 0 \;\Rightarrow\; r = 3$

$y(-2) = C(-2)^r = 8 \;\Rightarrow\; C \neq 0 \quad \therefore\; C(-2)^3 = 8 \Rightarrow C = -1$

16 (b). $-\infty < t < 0$ since $p(t) = \dfrac{-3}{t}$

16 (c). $y(t) = -t^3, \; -\infty < t < \infty$.

17. $y(t) = 0$ satisfies all of these conditions.

Section 2.2

1 (a). First, we will integrate $p(t) = 3$ to find $P(t) = 3t$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{-3t}$.

1 (b). $y(0) = C = -3$. Therefore, the solution to the initial value problem is $y = -3e^{-3t}$.

2 (a). $y' - \frac{1}{2}y = 0$, $(e^{-t/2}y)' = 0$, $y = Ce^{t/2}$.

2 (b). $y(-1) = Ce^{-1/2} = 2$, $C = 2e^{1/2}$ $y(t) = 2e^{(t+1)/2}$

3 (a). We can rewrite this equation into the conventional form: $y' - 2ty = 0$. Then we will integrate $p(t) = -2t$ to find $P(t) = -t^2$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{t^2}$.

3 (b). $y(1) = Ce = 3$. Solving for C yields $C = 3e^{-1}$. Therefore, the solution to the initial value problem is $y(t) = 3e^{-1}e^{t^2} = 3e^{(t^2-1)}$.

4 (a). $ty' - 4y = 0 \Rightarrow y' - \frac{4}{t}y = 0$. $\int -\frac{4}{t}\,dt = -4\ln|t| = -\ln(t^4)$ $\therefore$ $\mu = \frac{1}{t^4}$

$\frac{1}{t^4}y' - \frac{4}{t^5}y = (t^{-4}y)' = 0$ $y = Ct^4$.

4 (b). $y(1) = C = 1$ $\therefore$ $y(t) = t^4$.

5 (a). For this D.E., $p(t) = -3$. Integrating gives us $P(t) = -3t$. An integrating factor is, then, $\mu(t) = e^{-3t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{-3t}y' - 3e^{-3t}y = (e^{-3t}y)' = 6e^{-3t}$. Integrating both sides yields $e^{-3t}y = -2e^{-3t} + C$. Solving for y gives us $y = -2 + Ce^{3t}$.

5 (b). $y(0) = 1 = -2 + C$. Solving for C yields $C = 3$, and thus our final solution is $y = -2 + 3e^{3t}$.

6 (a). $y' - 2y = e^{3t}$, $y(0) = 3$. $(e^{-2t}y)' = e^t \Rightarrow e^{-2t}y = e^t + C \Rightarrow y = e^{3t} + Ce^{2t}$

6 (b). $y(0) = 1 + C = 3 \Rightarrow C = 2$, $y = e^{3t} + 2e^{2t}$.

7 (a). Putting this D.E. in the conventional form, we have $y' + \frac{3}{2}y = \frac{1}{2}e^t$. For this D.E., $p(t) = \frac{3}{2}$. Integrating gives us $P(t) = \frac{3}{2}t$. An integrating factor is, then, $\mu(t) = e^{\frac{3}{2}t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{\frac{3}{2}t}y' + \frac{3}{2}e^{\frac{3}{2}t}y = (e^{\frac{3}{2}t}y)' = \frac{1}{2}e^{\frac{5}{2}t}$. Integrating both sides yields $e^{\frac{3}{2}t}y = \frac{1}{5}e^{\frac{5}{2}t} + C$. Solving for y gives us $y = \frac{1}{5}e^t + Ce^{-\frac{3}{2}t}$.

7 (b). $y(0) = 0 = \frac{1}{5} + C$. Solving for C yields $C = -\frac{1}{5}$, and thus our final solution is $y = \frac{1}{5}e^{t} - \frac{1}{5}e^{-\frac{3}{2}t}$.

8 (a). $y' + y = 1 + 2e^{-t}\cos(2t),\ y(\pi/2) = 0 \quad \therefore\ (e^{t}y)' = e^{t} + 2\cos 2t$

$e^{t}y = e^{t} + \sin 2t + C \ \Rightarrow\ y = 1 + e^{-t}\sin 2t + Ce^{-t}$.

8 (b). $y(\pi/2) = 1 + Ce^{-\pi/2} = 0 \ \Rightarrow\ C = -e^{\pi/2};\ y = 1 + e^{-t}\sin 2t - e^{-(t-\pi/2)}$.

9 (a). Putting this D.E. in the conventional form, we have $y' + \frac{\cos(t)}{2}y = -\frac{3}{2}\cos(t)$. For this D.E., $p(t) = \frac{\cos(t)}{2}$. Integrating gives us $P(t) = \frac{\sin(t)}{2}$. An integrating factor is, then, $\mu(t) = e^{\frac{\sin(t)}{2}}$. Multiplying the D.E. by $\mu(t)$, we obtain

$e^{\frac{\sin(t)}{2}}y' + \frac{\cos(t)}{2}e^{\frac{\sin(t)}{2}}y = (e^{\frac{\sin(t)}{2}}y)' = -\frac{3\cos(t)}{2}e^{\frac{\sin(t)}{2}}$. Integrating both sides yields

$e^{\frac{\sin(t)}{2}}y = -3e^{\frac{\sin(t)}{2}} + C$. Solving for y gives us $y = -3 + Ce^{-\frac{\sin(t)}{2}}$.

9 (b). $y(0) = -4 = -3 + C$. Solving for C yields $C = -1$, and thus our final solution is $y = -3 - e^{-\frac{\sin(t)}{2}}$.

10 (a). $y' + 2y = e^{-t} + t + 1,\ y(-1) = e,\ (e^{2t}y)' = e^{t} + te^{2t} + e^{2t}$

$ye^{2t} = e^{t} + \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \frac{1}{2}e^{2t} + C \ \Rightarrow\ y = e^{-t} + \frac{t}{2} + \frac{1}{4} + Ce^{-2t}$.

10 (b). $y(-1) = e - \frac{1}{2} + \frac{1}{4} + Ce^{2} = e \ \Rightarrow\ C = \frac{1}{4}e^{-2}$

$\therefore\ y = e^{-t} + \frac{t}{2} + \frac{1}{4} + \frac{1}{4}e^{-2(t+1)}$.

11. We can rewrite this equation into the conventional form: $y' + \frac{4}{t}y = 0$. Then we will integrate $p(t) = \frac{4}{t}$ to find $P(t) = 4\ln|t| = \ln t^{4}$. The general solution, then, is

$y(t) = Ce^{-P(t)} = Ce^{-\ln t^{4}} = Ce^{\ln t^{-4}} = Ct^{-4}$.

12. $\mu = \exp(t - \cos t) \quad \therefore\ y(t) = Ce^{-(t-\cos t)}$.

13. First, we will integrate $p(t) = -2\cos(2t)$ to find $P(t) = -\sin(2t)$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{\sin(2t)}$.

14. $((t^{2}+1)y)' = 0\ y = \frac{C}{t^{2}+1}$.

15. We can rewrite this equation into the conventional form: $y' - 3(t^2+1)y = 0$. Then we will integrate $p(t) = -3(t^2+1)$ to find $P(t) = -t^3 - 3t$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{t^3+3t}$.

16. $y' + e^{-t}y = 0 \quad \therefore \int e^{-t}dt = -e^{-t} \; (-e^{e^{-t}} y)' = 0 \; y = Ce^{e^{-t}}$.

17. For this D.E., $p(t) = 2$. Integrating gives us $P(t) = 2t$. An integrating factor is, then, $\mu(t) = e^{2t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{2t}y' + 2e^{2t}y = (e^{2t}y)' = e^{2t}$. Integrating both sides yields $e^{2t}y = \frac{1}{2}e^{2t} + C$. Therefore, the general solution is $y(t) = \frac{1}{2} + Ce^{-2t}$.

18. $y' + 2y = e^{-t} \;\Rightarrow\; (e^{2t}y)' = e^t \;\Rightarrow\; e^{2t}y = e^t + C \;\Rightarrow\; y = e^{-t} + Ce^{-2t}$.

19. For this D.E., $p(t) = 2$. Integrating gives us $P(t) = 2t$. An integrating factor is, then, $\mu(t) = e^{2t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{2t}y' + 2e^{2t}y = (e^{2t}y)' = 1$. Integrating both sides yields $e^{2t}y = t + C$. Therefore, the general solution is $y(t) = te^{-2t} + Ce^{-2t}$.

20. $y' + 2ty = t \;\Rightarrow\; (e^{t^2}y)' = te^{t^2} \;\Rightarrow\; e^{t^2}y = \frac{1}{2}e^{t^2} + C \;\Rightarrow\; y = \frac{1}{2} + Ce^{-t^2}$.

21. Putting this equation into the conventional form gives us $y' + \frac{2}{t}y = t$. For this D.E., $p(t) = \frac{2}{t}$. Integrating gives us $P(t) = 2\ln t$. An integrating factor is, then, $\mu(t) = e^{\ln t^2} = t^2$. Multiplying the D.E. by $\mu(t)$, we obtain $t^2y' + 2ty = (t^2y)' = t^3$. Integrating both sides yields $t^2y = \frac{1}{4}t^4 + C$.

Therefore, the general solution is $y(t) = \frac{1}{4}t^2 + Ct^{-2}$.

22. $(t^2+4)y' + 2ty = t^2(t^2+4) \;\Rightarrow\; y' + \frac{2t}{t^2+4}y = t^2, \; \mu = e^{\ln(t^2+4)} = t^2+4$

$\therefore \; ((t^2+4)y)' = t^2(t^2+4) = t^4 + 4t^2 \;\Rightarrow\; (t^2+4)y = \frac{t^5}{5} + \frac{4t^3}{3} + C \; y = \frac{t^5/5 + 4t^3/3 + C}{(t^2+4)}$.

23. For this D.E., $p(t) = 1$. Integrating gives us $P(t) = t$. An integrating factor is, then, $\mu(t) = e^t$. Multiplying the D.E. by $\mu(t)$, we obtain $e^ty' + e^ty = (e^ty)' = te^t$. Integrating both sides yields $e^ty = te^t - e^t + C$. Therefore, the general solution is $y(t) = t - 1 + Ce^{-t}$.

24. $y' + 2y = \cos 3t \;\Rightarrow\; (e^{2t}y)' = e^{2t}\cos 3t$

$u = e^{2t} \; dv = \cos 3t dt$

$du = 2e^{2t}dt \; v = \frac{1}{3}\sin 3t \qquad \int e^{2t}\cos 3t dt = \frac{e^{2t}}{3}\sin 3t - \frac{2}{3}\int e^{2t}\sin 3t dt$

$u = e^{2t}\ dv = \sin 3t dt$

$du = 2e^{2t} dt\ v = -\frac{1}{3}\cos 3t \qquad \int e^{2t}\sin 3t dt = -\frac{e^{2t}}{3}\cos 3t + \frac{2}{3}\int e^{2t}\cos 3t dt$

$\therefore\ I = \frac{e^{2t}}{3}\sin 3t - \frac{2}{3}\left\{-\frac{e^{2t}}{3}\cos 3t + \frac{2}{3}I\right\} \Rightarrow I(1+\frac{4}{9}) = \frac{e^{2t}}{3}(\sin 3t + 2\cos 3t)$

$\therefore\ I = \frac{3}{13}e^{2t}(\sin 3t + 2\cos 3t)$

$\therefore\ e^{2t}y = \frac{3}{13}e^{2t}(\sin 3t + 2\cos 3t) + C \Rightarrow y = \frac{3}{13}(\sin 3t + 2\cos 3t) + Ce^{-2t}$

25 (a). #2

25 (b). #3

25 (c). #1

26. $y(t) = y_0 e^{-\alpha t}$ $4 = y_0 e^{-\alpha}$, $1 = y_0 e^{-3\alpha}$ Divide: $4 = e^{2\alpha} \Rightarrow \alpha = \frac{1}{2}\ln 4 = \ln 2$

and $y_0 = e^{3\alpha} = e^{\frac{3}{2}\ln 4} = e^{\ln(8)} = 8$. $\therefore\ y(t) = 8e^{-(\ln 2)t}$.

27. First, we should put the equation into our conventional form: $y' - \frac{\alpha}{t}y = 0$. Integrating $p(t) = -\frac{\alpha}{t}$ gives us $P(t) = -\alpha\ln|t| = \ln|t^{-\alpha}|$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{-\ln|t^{-\alpha}|} = Ce^{\ln|t^{\alpha}|} = Ct^{\alpha}$. Using the general solution and the point $(2,1)$, we can solve for C in terms of α: $y(2) = 1 = C\cdot 2^{\alpha}; C = 2^{-\alpha}$. We can then substitute this value for C into the general solution at the point $(4,4)$: $y(4) = 4 = 2^{-\alpha}\cdot 4^{\alpha} = 4^{-\alpha/2}\cdot 4^{\alpha} = 4^{\alpha/2}$. Setting the exponents equal to each other yields $1 = \frac{\alpha}{2}; \alpha = 2$. Finally, solving for y_0, $y_0 = y(1) = 2^{-2}\cdot 1^2 = \frac{1}{4}$.

28 (a). The general solution is $y = ce^{-t/2}$, so the corresponding graph is graph 2. $y(0) = 2$.

28 (b). The general solution is $y = ce^{-(\sin 2t)/2}$, so the corresponding graph is graph 4. $y(0) = 3$.

28 (c). The general solution is $y = ce^{0.1\left(t - \frac{\sin 2t}{2}\right)}$, so the corresponding graph is graph 1. $y(0) = 1$.

28 (d). The general solution is $y = ce^{t/10}$, so the corresponding graph is graph 3. $y(0) = 2$.

29 (a). $B(c) = A(c) - A^*$ and differentiating gives us $\frac{dB}{dc} = \frac{dA}{dc} = -kB$, $B(0) = -A^*$.

29 (b). $B(c) = -A^*e^{-kc} = A(c) - A^*$, and substitution gives us $A(c) = A^*(1 - e^{-kc})$. The activity does not ever exceed A^* because $A(c)$ only approaches the value of A^* as c approaches ∞. Alternatively, the value of $\left(1 - e^{-kc}\right)$ is never higher than 1.

29 (c). Substituting the condition into our given equation, we have $0.95A^* = A^*(1 - e^{-kc})$. Simplification gives us $-0.05 = -e^{-kc} \Rightarrow -kc = \ln(1/20) = -\ln(20)$, and solving for c yields $c_{0.95} = \frac{1}{k}\ln(20)$.

30. $y' + \frac{4}{t}y = \alpha t,\ \mu = t^4$

$$t^4y' + 4t^3y = \alpha t^5 = (t^4y)' \Rightarrow t^4y = \alpha\frac{t^6}{6} + C \Rightarrow y = \frac{\alpha t^2}{6} + Ct^{-4}$$

$$y(1) = -\frac{1}{3} = \frac{\alpha}{6} + C \Rightarrow C = -\frac{1}{3} - \frac{\alpha}{6} \equiv 0 \Rightarrow \alpha = -2,\ y = -\frac{t^2}{3}.$$

31. Multiplying both sides of the equation by the integrating factor, $\mu(t) = e^{2t}$, we have $e^{2t}y = e^{2t}(Ce^{-2t} + t + 1) = e^{2t}(t+1) + C$. Differentiating gives us $(e^{2t}y)' = e^{2t}(1) + 2e^{2t}(t+1) = e^{2t}(2t+3)$. Therefore, $(e^{2t}y)' = (\mu(t)y)' = \mu(t)\cdot g(t) = e^{2t}(2t+3) \Rightarrow g(t) = 2t + 3$ and $\mu(t) = e^{2t} = e^{P(t)} \Rightarrow P(t) = 2t \Rightarrow p(t) = 2$.

32. $2tCe^{t^2} + pCe^{t^2} = 0 \Rightarrow p(t) = -2t$. Substituting, $(Ce^{t^2} + 2)' - 2t(Ce^{t^2} + 2) = -4t \Rightarrow g(t) = -4t$.

33. Multiplying both sides of the equation by the integrating factor, $\mu(t) = t$, we have $ty = t(Ct^{-1} + 1) = t + C$. Differentiating gives us $(ty)' = 1$. Therefore, $(ty)' = (\mu(t)y)' = \mu(t)\cdot g(t) = 1 = (t)(t^{-1}) \Rightarrow g(t) = t^{-1}$ and $\mu(t) = t = e^{P(t)} \Rightarrow P(t) = \ln t \Rightarrow p(t) = \frac{1}{t} = t^{-1}$.

34. $(e^{-t} + t - 1)' + (e^{-t} + t - 1) = t \Rightarrow g(t) = t,\ y_0 = 0$.

35. $y(t) = -2e^{-t} + e^t + \sin t \Rightarrow y_0 = y(0) = -2 + 1 + 0 = -1$.

If $y(t) = -2e^{-t} + e^t + \sin t$, then $y' = 2e^{-t} + e^t + \cos t$.

Substituting in $y' + y = g(t)$, $(2e^{-t} + e^t + \cos t) + (-2e^{-t} + e^t + \sin t) = 2e^t + \cos t + \sin t = g(t)$.

36. $y' + (1 + \cos t)y = 1 + \cos t,\ y(0) = 3,\ \mu = e^{t + \sin t}$.

$$(e^{t+\sin t}y)' = (1 + \cos t)e^{t+\sin t} = (e^{t+\sin t})' \Rightarrow e^{t+\sin t}y = e^{t+\sin t} + C \Rightarrow y = 1 + Ce^{-(t+\sin t)}.$$

$$y(0) = 1 + C = 3 \Rightarrow C = 2 \ \therefore\ y = 1 + 2e^{-(t+\sin t)} \text{ and } \lim_{t\to\infty} y(t) = 1.$$

37. Putting this D.E. in the conventional form, we have $y' + 2y = e^{-t} - 2$. For this D.E., $p(t) = 2$. An integrating factor is, then, $\mu(t) = e^{2t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{2t}y' + 2e^{2t}y = (e^{2t}y)' = e^{t} - 2e^{2t}$. Integrating both sides yields $e^{2t}y = e^{t} - e^{2t} + C$. Solving for y gives us $y = e^{-t} - 1 + Ce^{-2t}$, and with our initial condition, $y(0) = -2 = 1 - 1 + C$. Solving for C yields $C = -2$, and thus our final solution is $y = e^{-t} - 1 - 2e^{-2t}$.
Therefore, $\lim_{t\to\infty} y(t) = -1$.

38. $y = ce^{-t} + te^{-t} \Rightarrow y_0 = c$. $y'(t) = -y_0e^{-t} + e^{-t} - te^{-t}$. $y'(1) = 0 \Rightarrow (-y_0 + 1 - 1)e^{-1} = 0 \therefore y_0 = 0$.

39. The general solution of the D.E. is $y = Ce^{-\lambda t} + \frac{1}{\lambda}$, $\lambda \neq 0$; $y = t + C$, $\lambda = 0$. Therefore, the relevant limits are: $\lim_{t\to\infty} y$ does not exist for $\lambda = 0$ and $\lambda < 0$; $\lim_{t\to\infty} y = C \cdot 0 + \frac{1}{\lambda} = \frac{1}{\lambda}$ for $\lambda > 0$.

40. On $[1,2]$:
$y' + \frac{1}{t}y = 3t,\ y(1) = 1$. An integrating factor is $\mu(t) = t$. Multiplying the D.E. by $\mu(t)$, we obtain $(ty)' = 3t^2 \Rightarrow ty = t^3 + C \Rightarrow y = t^2 + Ct^{-1},\ y(1) = 1 + C = 1 \Rightarrow C = 0$. Therefore, the solution for $1 \le t \le 2$ is $y = t^2$ and $y(2) = 4$.
On $[2,3]$:
$y' + \frac{1}{t}y = 0,\ y(2) = 4$. An integrating factor is $\mu(t) = t$. Multiplying the D.E. by $\mu(t)$, we obtain $(ty)' = 0 \Rightarrow ty = C \Rightarrow y = Ct^{-1},\ y(2) = \frac{C}{2} = 4 \Rightarrow C = 8$. Therefore, the solution for $2 \le t \le 3$ is $y = \frac{8}{t}$.

41. On $[0,\pi]$:
$y' + (\sin t)y = \sin t,\ y(0) = 3$. An integrating factor is $\mu(t) = e^{-\cos t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{-\cos t}y' + e^{-\cos t}(\sin t)y = (e^{-\cos t}y)' = (\sin t)e^{-\cos t}$. Integrating both sides yields $e^{-\cos t}y = e^{-\cos t} + C$. Solving for y gives us $y = 1 + Ce^{\cos t}$, and with our initial condition, $y(0) = 3 = 1 + Ce \Rightarrow C = 2e^{-1}$. Therefore, the solution for $0 \le t \le \pi$ is $y = 1 + 2e^{\cos t - 1}$ and $y(\pi) = 1 + 2e^{-2}$.
On $[\pi,2\pi]$:
$y' + (\sin t)y = -\sin t,\ y(\pi) = 1 + 2e^{-2}$. Multiplying the D.E. by $\mu(t) = e^{-\cos t}$, we obtain $e^{-\cos t}y' + e^{-\cos t}(\sin t)y = (e^{-\cos t}y)' = (-\sin t)e^{-\cos t}$.

Integrating both sides yields $e^{-\cos t}y = -e^{-\cos t} + C$. Solving for y gives us $y = -1 + Ce^{\cos t}$, and with our initial condition, $y(\pi) = 1 + 2e^{-2} = -1 + Ce^{-1} \Rightarrow C = 2e^{1} + 2e^{-1}$. Therefore, the solution for $\pi \le t \le 2\pi$ is $y = -1 + 2e^{\cos t + 1} + 2e^{\cos t - 1}$.

42. On $[0,1]$: $y' = 2,\ y(0) = 1$.

$y = 2t + C,\ y(0) = C = 1 \Rightarrow C = 1$.

Therefore, the solution for $0 \le t \le 1$ is $y = 2t + 1$ and $y(1) = 3$.

On $[1,2]$: $y' + \dfrac{1}{t}y = 2,\ y(1) = 3$. An integrating factor is $\mu(t) = t$. Multiplying the D.E. by $\mu(t)$, we obtain $(ty)' = 2t \Rightarrow ty = t^2 + C \Rightarrow y = t + Ct^{-1},\ y(1) = 1 + C = 3 \Rightarrow C = 2$. Therefore, the solution for $1 \le t \le 2$ is $y = t + \dfrac{2}{t}$.

43. On $[0,1]$:

$y' + (2t-1)y = 0,\ y(0) = 3$. An integrating factor is $\mu(t) = e^{t^2 - t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{t^2-t}y' + e^{t^2-t}(2t-1)y = (e^{t^2-t}y)' = 0$. Integrating both sides yields $e^{t^2-t}y = C$. Solving for y gives us $y = Ce^{t-t^2}$, and with our initial condition, $y(0) = 3 = C$. Therefore, the solution for $0 \le t \le 1$ is $y = 3e^{t-t^2}$ and $y(1) = 3$.

On $[1,3]$:

$y' + (0)y = y' = 0,\ y(1) = 3$. Integrating gives us $y = C = 3$. Therefore, the solution for $1 \le t \le 3$ is $y = 3$ and $y(3) = 3$.

On $[3,4]$:

$y' + \left(-\frac{1}{t}\right)y = 0,\ y(3) = 3$. An integrating factor is $\mu(t) = e^{-\ln t} = \frac{1}{t}$. Multiplying the D.E. by $\mu(t)$, we obtain $\frac{1}{t}y' - \frac{1}{t^2}y = (\frac{1}{t}y)' = 0$. Integrating both sides yields $\frac{1}{t}y = C$. Solving for y gives us $y = Ct$, and with our initial condition, $y(3) = 3 = C(3) \Rightarrow C = 1$. Therefore, the solution for $3 \le t \le 4$ is $y = t$.

44. $y(t) = t\{Si(t) - Si(1) + 3\}$

Section 2.3

1 (a). To begin, $Q(0) = 0$ and $Q' = (0.2)(3) - \dfrac{Q}{100}(3)$. Putting the second equation in the conventional form, we have $Q' + 0.03Q = 0.6$. Multiplying both sides of this equation by the integrating factor $\mu(t) = e^{0.03t}$ gives us $(e^{0.03t}Q)' = 0.6e^{0.03t}$. Integrating both sides

yields $e^{0.03t}Q = 0.6 \cdot \frac{100}{3} e^{0.03t} + C = 20e^{0.03t} + C$. Solving for Q, we have $Q = 20 + Ce^{-0.03t}$. $Q(0) = 0 = 20 + C$, so $C = -20$. With this value for C, our final equation for Q is $Q = 20(1 - e^{-0.03t})$. Thus, $Q(10) = 20(1 - e^{-0.3}) \approx 5.18$ lb.

1 (b). $\lim_{t\to\infty} Q(t) = 20$ lb and the limiting concentration is 0.2 lb/gal.

2. $Q' = -\frac{Q}{500} \cdot 10, ?Q(0) = 50$. $\frac{Q'}{Q} = -\frac{1}{50} \Rightarrow Q = Ce^{-t/50}, ?C = 50$. $Q(t) = 50e^{-t/50}$.

$50e^{-t/50} = 5 \Rightarrow t = 115.129$ min. $= 1.9$ hours.

3. First, $V = 100(70)(20) = 140{,}000 m^3$. Substituting into equation (2), we have $Q' = 0 - \frac{Q}{v} r$, and so the general solution is $Q = Q_0 e^{-\frac{r}{v}t}$. Using the given condition, we obtain $0.01Q_0 = Q_0 e^{-\frac{r}{v}30}$, and solving for r yields $-\frac{r}{v} = \frac{1}{30}\ln(0.01) \Rightarrow r = \frac{v}{30}\ln(100)$.

Substituting the known volume, $r = \frac{140{,}000}{30}\ln(100) \approx 21{,}491\ \mathrm{m^3/min}$. Finally, the fraction of the volume of air that must be vented is $\frac{r}{v} = \frac{1}{30}\ln(100) = 0.1535 \quad (\approx 15.4\%)$.

4 (a). $Q(0) = 5$, $Q' = 0.1r - \frac{Q}{200} r$. $(e^{0.005rt}Q)' = 0.1re^{0.005rt}$.

$e^{0.005rt}Q = 0.1(200)e^{0.005rt} + C = 20e^{0.005rt} + C$. $Q = 20 + Ce^{-0.005rt}$.

$Q(0) = 5 = 20 + C \Rightarrow C = -15$. $Q = 20 - 15e^{-0.005rt}$. $Q(20) = 15 = 20 - 15e^{-\frac{20}{200}r} \Rightarrow$

$r = 10.99$ gal/min.

4 (b). It would not be possible.

5 (a). Substituting into equation (2), we have $Q' = (10te^{-t/50})(100) - \frac{Q}{5000}(100)$, $Q(0) = 0$.

Simplification gives us $Q' = -\frac{1}{50}Q + 1000te^{-t/50}$, and so $(Qe^{t/50})' = 1000t$.

$Qe^{t/50} = 500t^2 + C$, and so the general solution is $Q = 500t^2e^{-t/50} + Ce^{-t/50}$. With the initial condition of $Q(0) = C = 0$, $Q(t) = 500t^2e^{-t/50}$ mg.

5 (b). $Q' = 500\left(2t - \frac{t^2}{50}\right)e^{-t/50} = 0 \Rightarrow t^2 = 100t$, and solving for t gives us $t = 100$ minutes.

Substituting this time back into our equation for Q, we obtain the maximum concentration:

$\frac{Q(100)}{5000} = \frac{500(100)^2}{5000}e^{-2} = 1000e^{-2} \approx 135.3\ \mathrm{mg/gal}$.

5 (c). To find the two times at which the concentration is $100\,{}^{\text{mg}}\!/_{\text{gal}}$, we must examine a graph of Q.

Yes, the dosing was effective.

6 (a). $V(t) = 400 + t - 2t \Rightarrow t = 400$ min.

6 (b). $V(300) = 100$. $Q'(t) = 0.1 - \dfrac{2Q}{400-t}$,? $Q(0) = 0$. $Q' + \dfrac{2Q}{400-t} = 0.1$.

$$\mu(t) = e^{\int \frac{2}{400-t}dt} = (400-t)^{-2} \Rightarrow \left[\frac{Q}{(400-t)^2}\right]' = \frac{0.1}{(400-t)^2}.$$

$$\frac{Q}{(400-t)^2} = \frac{0.1}{400-t} + C \Rightarrow Q = 0.1(400-t) + C(400-t)^2.$$

$$Q(0) = 0 \Rightarrow 40 + 160000C = 0 \Rightarrow C = -\frac{1}{4000} \Rightarrow Q(t) = 0.1(400-t) - \frac{1}{4000}(400-t)^2.$$

$Q(300) = 7.5$ lbs.

6 (c). $Q'(t) = -0.1 + \dfrac{1}{2000}(400-t) = 0 \Rightarrow t = 200$. $Q(200) = 20 - \dfrac{1}{4000}(200)^2 = 10$ lbs.

7 (a). To begin, $Q(0) = 10$, $V(0) = 100$, and $V(t) = 100 + t$. Since the tank has a capacity of 700 gallons, $100 + t = 700$. Solving for t yields $t = 600$ minutes.

7 (b). $Q' = (0.5)(3) - \dfrac{Q}{100+t}(2)$. Putting this in the conventional form, we have $Q' + \dfrac{2}{100+t}Q = \dfrac{3}{2}$.

Multiplying both sides of the equation by the integrating factor $\mu(t) = e^{2\ln(100+t)} = (100+t)^2$

gives us $((100+t)^2 Q)' = \dfrac{3}{2}(100+t)^2$. Integrating both sides yields $(100+t)^2 Q = \dfrac{(100+t)^3}{2} + C$,

and solving for Q, we have $Q = \dfrac{100+t}{2} + \dfrac{C}{(100+t)^2}$. $Q(0) = 10 = 50 + \dfrac{C}{100^2}$, and solving for C

yields $C = -40(100)^2 = -400{,}000$.

Substituting this value of C back into our equation for Q gives us our final equation

for Q, $Q(t) = \dfrac{100+t}{2} - \dfrac{400{,}000}{(100+t)^2}$. $V(t) = 400$ at $t = 300$, so

$Q(300) = \dfrac{400}{2} - \dfrac{400{,}000}{(400)^2} = 197.5$ lb. The concentration, then, is $\dfrac{197.5}{400}$ lb/gal.

7 (c). $Q(600) = \dfrac{700}{2} - \dfrac{400{,}000}{(700)^2} \approx 349.2$lb. The concentration, then, is $\dfrac{349.2}{700} \approx .4988$ lb/gal.

8 (a). $\dfrac{Q(t)}{1000} = \dfrac{e^{-t/500}}{50} \Rightarrow Q(t) = 20e^{-t/500} \Rightarrow Q(0) = 20$lb.

8 (b). $-\dfrac{20}{500}e^{-t/500} + \dfrac{e^{-t/500}}{25} = 2c_i(t) \Rightarrow c_i = 0$

9 (a). $\dfrac{Q(t)}{1000} = \dfrac{1}{20}\left[1 - e^{-t/500}\right]$ can be simplified to $Q(t) = 50\left[1 - e^{-t/500}\right]$, and so $Q(0) = 0$.

9 (b). Differentiation gives us $Q'(t) = \frac{1}{10}e^{-t/500}$, and substitution gives us $Q'(t) + 2c_0(t) = 2c_i(t) = \frac{1}{10}e^{-t/500} + \frac{1}{10}\left[1 - e^{-t/500}\right]$. Solving for $c_i(t)$ gives us $c_i(t) = \frac{1}{20}\ \text{lb}/\text{gal}$.

10 (a). $\frac{Q(t)}{1000} = \frac{te^{-t/500}}{500} \Rightarrow Q(t) = 2te^{-t/500} \Rightarrow Q(0) = 0$.

10 (b). $Q'(t) = 2e^{-t/500} - \frac{2t}{500}e^{-t/500} \Rightarrow 2e^{-t/500}\left[1 - \frac{t}{500}\right] + \frac{2te^{-t/500}}{500} = 2c_i(t) \Rightarrow c_i(t) = e^{-t/500}\ \text{lb/gal}$.

11 (a). $Q' = \alpha\frac{Q}{500}(15) - \frac{Q}{500}(15)$

11 (b). Our boundary condition is $Q(180) = 0.01Q_0$. Substitution gives us $Q' = \frac{-(1-\alpha)}{500}(15)Q$, and so $Q = Q_0e^{-.03(1-\alpha)t}$. Our condition gives us

$.01 = e^{-.03(1-\alpha)(180)}$, which we can simplify to $e^{-5.4(1-\alpha)} = .01$. Solving for α,

$5.4(1-\alpha) = \ln(100) \;\Rightarrow\; 1-\alpha = 0.8528 \;\Rightarrow\; \alpha = 0.1472$.

12 (a). $Q' = 4r - (10+r)\frac{Q}{2},\quad Q(0) = 0$

12 (b). $Q_e = (1\,oz/gal)\cdot(2gal) = 2oz$. Therefore, $4r - \frac{10+r}{2}\cdot 2 = 0 \Rightarrow r = \frac{10}{3}\text{gal/min}$.

12 (c). $Q' + \frac{10+r}{2}Q = 4r \Rightarrow Q = \frac{8r}{10+r} + Ce^{-(10+r)t/2}$

$Q(0) = 0 \Rightarrow C = \frac{-8r}{10+r}$ and $Q = \frac{8r}{10+r}\left(1 - e^{-(10+r)t/2}\right)$

With $r = \frac{10}{3}, \frac{8r}{10+r} = 2$. Set $1.98 = 2\left(1 - e^{-(10+10/3)t/2}\right)$. Therefore, $t = 0.69077...\text{min}$.

13 (a). $Q_A(0) = 1000$, $Q_B(0) = 0$, $Q_A{}' = 0 - 1000\left(\frac{Q_A}{500{,}000}\right)$,

and $Q_B{}' = 1000\left(\frac{Q_A}{500{,}000}\right) - 1000\left(\frac{Q_B}{200{,}000}\right)$.

13 (b). Putting the equation for $Q_A{}'$ into the conventional form, we have $Q_A{}' = -\frac{1}{500}Q_A$.

Thus, $Q_A = 1000e^{-\frac{t}{500}}$. Putting the equation for $Q_B{}'$ into the conventional form, we have $Q_B{}' + \frac{1}{200}Q_B = 2e^{-\frac{t}{500}}$. Multiplying both sides by the integrating factor $\mu(t) = e^{\frac{t}{200}}$

yields $(Q_B e^{\frac{t}{200}})' = 2e^{t\left(\frac{1}{200}-\frac{1}{500}\right)} = 2e^{\frac{3t}{1000}}$. Integrating both sides gives us $Q_B e^{\frac{t}{200}} = \frac{2000}{3}e^{\frac{3t}{1000}} + C$, and solving for Q_B, $Q_B = \frac{2000}{3}e^{-\frac{t}{500}} + Ce^{-\frac{t}{200}}$. $Q_B(0) = 0 = \frac{2000}{3} + C$, so $C = -\frac{2000}{3}$. Substituting this value back into our equation, we have $Q_B = \left(\frac{2000}{3}\right)\left(e^{-\frac{t}{500}} - e^{-\frac{t}{200}}\right)$

13 (c). Setting $Q_B{}' = 0$, we have $0 = \left(\frac{2000}{3}\right)\left(-\frac{1}{500}e^{-\frac{t}{500}} + \frac{1}{200}e^{-\frac{t}{200}}\right)$. Since $e^{-\frac{t}{500}+\frac{t}{200}} = \frac{500}{200}$,

$\frac{3}{1000}t = \ln\left(\frac{5}{2}\right)$, and thus $t = \frac{1000}{3}\ln\left(\frac{5}{2}\right) \approx 305.4$ hours.

13 (d). Here, we want to determine t_A such that $Q_A(t_A) = \frac{1}{2}$ lb and t_B such that $Q_B(t) \le 0.2$ lb where $t \le t_B$. This can be solved via plotting: $t_A \approx 3800$ hours and $t_B \approx 4056$ hours. Therefore, $t \approx 4056$ hours.

14 (a). $r_i = r_0 = 3 + \sin t \;\Rightarrow\; V = \text{constant}$.

14 (b). Expect $\lim_{t\to\infty} Q(t) = .5(200) = 100$ lb.

The tank is being "flushed out", albeit in a pulsating manner.

14 (c). $Q' = .5(3+\sin t) - \frac{Q}{200}(3+\sin t),\; Q(0) = 10$

$$Q' + \frac{3+\sin t}{200}Q = \frac{1}{2}(3+\sin t) \;\Rightarrow\; (Qe^{(3t-\cos t)/200})' = \frac{1}{2}(3+\sin t)e^{(3t-\cos t)/200}$$

$$Qe^{(3t-\cos t)/200} = 100e^{3t-\cos t} + C \;\Rightarrow\; Q = 100 + Ce^{-(3t-\cos t)/200}$$

$$Q(0) = 10 = 100 + Ce^{1/200} \;\Rightarrow\; C = -90e^{-1/200} \Rightarrow\; Q(t) = 100 - 90e^{-(3t-\cos t+1)/200}.$$

14 (d). $\lim_{t\to\infty} e^{-(3t-\cos t+1)/200} = 0 \;\Rightarrow\; \lim_{t\to\infty} Q(t) = 100$ lb.

15 (a). We do not expect the limit to exist since we do not expect concentration to stabilize.

15 (b). Substituting the relevant values into equation (2), we have: $Q' = .2(1+\sin t)(3) - \frac{Q}{200}(3),\; Q(0) = 10$.

15 (c). Simplifying the D.E. above, we have $Q' + \frac{3}{200}Q = 0.6(1+\sin t)$. Multiplying both sides by the integrating factor $\mu(t) = e^{(3/200)t}$, we obtain $\left(e^{(3/200)t}Q\right)' = 0.6e^{(3/200)t}(1+\sin t)$.

We then integrate as follows:

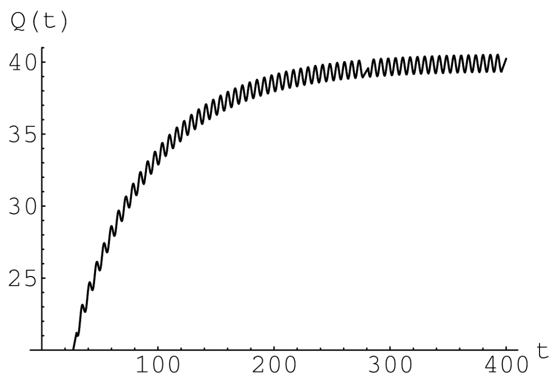
$$\int e^{at}\sin t\,dt = e^{at}\frac{(-\cos t + a\sin t)}{(1+a^2)} \Rightarrow e^{(3/200)t}Q = 0.6\left\{\frac{200}{3}e^{(3/200)t} + \frac{e^{(3/200)t}(-\cos t + \frac{3}{200}\sin t)}{1+(3/200)^2}\right\} + C.$$

Thus the general solution is:

$Q(t) = 0.6\left\{\frac{200}{3} + \frac{(-\cos t + \frac{3}{200}\sin t)}{1+(3/200)^2}\right\} + Ce^{-3/200 t} = 40 + \frac{-0.6\cos t + 0.009\sin t}{1.000225} + Ce^{-3/200 t}$. Using the initial condition, we have $Q(0) = 10 = 40 - \frac{0.6}{1.000225} + C$ and solving for C gives us $C = -30 + \frac{0.6}{1.000225}$. Thus, the solution to the initial value problem is:

$$Q(t) = 40 - 30e^{-3/200 t} + \left(\frac{0.6(e^{-3/200 t} - \cos t) + 0.009\sin t)}{1.000225}\right).$$

15 (d). The graph behaves as we would expect:



16. $\theta' = k(S-\theta)$, $S = 72$, $\theta(0) = 350$, $\theta(10) = 290$

$$\theta' + k\theta = kS \Rightarrow (e^{kt}\theta)' = ke^{kt}S \Rightarrow e^{kt}\theta = e^{kt}S + C \Rightarrow \theta = S + Ce^{-kt}$$

$$\theta(0) = \theta_0 = S + C \Rightarrow C = \theta_0 - S \Rightarrow \theta = S + (\theta_0 - S)e^{-kt}$$

$$290 = 72 + (350-72)e^{-k(10)} \Rightarrow 218 = 278e^{-10k},\ 10k = \ln\left(\frac{278}{218}\right)$$

$$k = \frac{1}{10}\ln\left(\frac{278}{218}\right);\ 120 = 72 + (350-72)e^{-kt} \Rightarrow e^{-kt} = \frac{48}{278}$$

$$t = -\frac{1}{k}\ln\left(\frac{48}{278}\right) = \frac{10\ln\left(\frac{278}{48}\right)}{\ln\left(\frac{278}{218}\right)} = \frac{10(1.756)}{0.243} \approx 72.2 \text{ min.}$$

17. From Newton's Law of Cooling, $\theta = S_0 + (\theta_0 - S_0)e^{-kt}, \theta(0) = 70$. Substitution gives us $\theta = 300 - 230e^{-kt}$. With our boundary condition of $\theta(10) = 150$, we obtain $150 = 300 - 230e^{-10k} \Rightarrow k = -0.1\ln\left(15/23\right)$.

With the value of *k* obtained, we can find the necessary value for S_0 to raise the object's temperature from 70 to 150 degrees in 5 minutes.

$\theta(5) = 150 = S_0 + (70 - S_0)e^{0.5\ln(15/23)} = S_0 + (70 - S_0)\left(\frac{15}{23}\right)^{0.5}$.

Thus $S_0 + 56.5301 - 0.8076S_0 = 150 \Rightarrow S_0 \approx 485.8$ degrees.

18. $\theta(t) = S_0 + (\theta_0 - S_0)e^{-kt}, \theta(0) = 150, \theta(2) = 100, \theta(4) = 90$

$\Rightarrow Q(t) = S_0 + (150 - S_0)e^{-kt}$. Solving simultaneous equations,

$100 = S_0 + (150 - S_0)e^{-2k}$ and $90 = S_0 + (150 - S_0)e^{-4k}$, we have

$$\left(\frac{100 - S_0}{150 - S_0}\right)^2 = \frac{90 - S_0}{150 - S_0} \Rightarrow S_0 = 87.5^0 F$$

19. $\theta(t) = 70 + 270e^{-t}$, and since the solution of (5) is $\theta(t) = S_0 + (\theta_0 - S_0)e^{-kt}$, we have $S_0 = 70^\circ\text{F}$ and $(\theta_0 - S_0) = 270 \Rightarrow \theta_0 = 340^\circ\text{F}$.

20. $\theta(t) = 390e^{-t/2}$, and since the solution of (5) is $\theta(t) = S_0 + (\theta_0 - S_0)e^{-kt}$, we have $S_0 = 0^\circ\text{F}$ and $(\theta_0 - S_0) = 390 \Rightarrow \theta_0 = 390^\circ\text{F}$.

21. $\theta(t) = 80 - 40e^{-2t}$, and since the solution of (5) is $\theta(t) = S_0 + (\theta_0 - S_0)e^{-kt}$, we have $S_0 = 80^\circ\text{F}$ and $(\theta_0 - S_0) = -40 \Rightarrow \theta_0 = 40^\circ\text{F}$.

22. To begin,

$\theta = S + (\theta_0 - S)e^{-kt}$. $120 = 350 + (40 - 350)e^{-10k}$. $k = -\frac{1}{10}\ln\left(\frac{350 - 120}{350 - 40}\right) \approx .02985$.

$\theta(20) = 350 + (40 - 350)e^{-20k} = 350 - (310)(0.550) \approx 179.5$ degrees.

$\theta(t) = 110 = 72 + (179.5 - 72)e^{-0.02985t}$. $t \approx -\frac{1}{0.02985}\ln\left(\frac{110 - 72}{179.5 - 72}\right) \approx 34.8$ minutes.

23. For the first cup, $\theta_1 = 72 + (34 - 72)e^{-kt}$. Thus, with the proper substitutions, $53 = 72 - 38e^{-kt_1}$. e^{-kt_1}, then, is equal to $\frac{19}{38}$. For the second cup, $\theta_2 = 34 + (72 - 34)e^{-kt}$. With the proper substitutions, we have $53 = 34 + 38e^{-kt_2}$. e^{-kt_2}, then, is equal to $\frac{19}{38}$. Thus, the two times are equal.

Section 2.4

1. $11{,}000{,}000 = 10{,}000{,}000e^{5k}$. Solving for k yields

$k = \frac{1}{5}\ln\left(\frac{11}{10}\right)$. $P(30) = 10{,}000{,}000e^{\frac{1}{5}\ln\left(\frac{11}{10}\right)(30)} = 10{,}000{,}000e^{\ln\left(\frac{11}{10}\right)^6} = 17{,}715{,}610$.

2. $2 = e^{kt} \Rightarrow t = \frac{\ln 2}{k} = 5\frac{\ln 2}{\ln\frac{11}{10}} \approx 36.36$ days.

3. Substitution gives us $1.3 = e^{2k}$, and solving for k, we have $k = \frac{1}{2}\ln(1.3)$. Substitution again gives us $3 = e^{kt}$, and solving for t yields $t = \frac{\ln 3}{k} = \frac{2\ln(3)}{\ln(1.3)} \approx 8.375$ wks.

4. $80{,}000 = 100{,}000e^{6k} \Rightarrow k = \frac{1}{6}\ln(.8) \Rightarrow (80{,}000 + 50{,}000)e^{\ln(0.8)} = 130{,}000 \cdot 0.8 = 104{,}000$.

5. $Q(t) = 100e^{-kt}, Q(0) = 100, Q(3) = 75$. Substitution gives us $\frac{75}{100} = e^{-3k}$, and solving for k, we have $k = -\frac{\ln\left(3/4\right)}{3}$. Thus $30 = 100\left(3/4\right)^{t/3}$, and solving for t gives us $t = \frac{3\ln\left(3/10\right)}{\ln\left(3/4\right)} \approx 12.56$ days, so it takes an additional 9.56 days for the material to reduce to 30 grams.

6. $Q(t) = Q_0e^{-kt}, Q(90) = 0.8Q_0 \Rightarrow -90k = \ln(0.8) \Rightarrow k = -\frac{\ln(0.8)}{90}$.

$\tau = \frac{\ln 2}{k} = \frac{-90\ln 2}{\ln(0.8)} \approx 279.56$ days.

7. $k = \frac{\ln 2}{\tau} = \frac{\ln 2}{2}$. Thus $Q(t) = Q_0e^{-\left(\ln 2/2\right)t}$. From our boundary condition, we have $Q(5) = Q_0 \cdot e^{-5\ln 2/2} = 20$, and solving for Q_0 gives us $Q_0 = 20e^{5\ln 2/2} \approx 113.137$ grams.

8 (a). $Q(30) = Ce^{-30k} = 100$, $Q(120) = Ce^{-120k} = 30 \Rightarrow C = 30e^{120k} \Rightarrow \frac{10}{3} = e^{90k}$

$\Rightarrow k = \frac{1}{90}\ln\left(\frac{10}{3}\right) \approx 0.01338$. $C = Q_0 = 149.4$mg.

8 (b). $\tau = \frac{\ln 2}{k} \approx 51.8$ days.

8 (c). $0.01 = e^{-kt}$. Solving for t, we have $t = -\frac{\ln(0.01)}{k} \approx 344.2$ days.

9. From the equation that models radioactive decay, we have $Q_A = 100e^{-kt}, k = \frac{\ln 2}{30}, Q_B = 50e^{-\lambda t}, \lambda = \frac{\ln 2}{90}$. Since we are looking for the time at which the populations are equal, we set $100e^{-kt} = 50e^{-\lambda t}$ and solve for t:

$$2 = e^{(k-\lambda)t} \Rightarrow \ln 2 = t(k-\lambda) \Rightarrow t = \frac{90}{2} \cdot \frac{\ln 2}{\ln 2} = 45 \text{ days}.$$

10 (a). $P' = kP + M,\ P(0) = P_0\ P' - kP = M,\ (e^{-kt}P)' = Me^{-kt}$

$$e^{-kt}P = -\frac{M}{k}e^{-kt} + C \ \Rightarrow\ P = -\frac{M}{k} + Ce^{kt},\ P_0 = -\frac{M}{k} + C$$

$$\therefore\ P(t) = -\frac{M}{k} + (P_0 + \frac{M}{k})e^{kt}$$

10 (b). $P_0 = -\frac{M}{k}.P_0$ and P must be nonnegative $\Rightarrow -\frac{M}{k} \geq 0$. If net immigration rate $M > 0$, net growth rate $k < 0$ and vice versa.

11 (a). For Strategy I, we have $M_I = kP_0$. For Strategy II, we have $M_{II} = P_0(e^k - 1)$.

11 (b). The net profit for each strategy would equal $(M)(^{profit}/_{fish})$, and so the profit for Strategy I is, then: $\Pr_I = 500{,}000(.3172)(.75) = 118{,}950$, and the profit for Strategy II is: $\Pr_{II} = 500{,}000(e^{.3172} - 1)(0.6) \approx 111{,}983$. Strategy I would be more profitable for the farm.

12 (a). $P_1(1) = -\frac{M}{k} + (P_0 + \frac{M}{k})e^k,\ P_1(2) = P_1(1)e^k = -\frac{M}{k}e^k + (P_0 + \frac{M}{k})e^{2k}$

$$P_2(1) = P_0e^k,\ P_2(2) = -\frac{M}{k} + (P_0e^k + \frac{M}{k})e^k$$

12 (b). $P_1(2) - P_2(2) = -\frac{M}{k}e^k + P_0e^{2k} + \frac{M}{k}e^{2k} + \frac{M}{k} - P_0e^{2k} - \frac{M}{k}e^k = \frac{M}{k}(e^{2k} - 2e^k + 1)$

$= \frac{M}{k}(e^k - 1)^2$.Since $M > 0$, $P_1(2) > P_2(2)$ if $k > 0$ and $P_1(2) < P_2(2)$ if $k < 0$.

12 (c). If $k > 0$, introduce the immigrants as early as possible. If $k < 0$, introduce as late as possible.

13 (a). $\tau = \frac{\ln 2}{k} = 5730 \ \Rightarrow\ k = \frac{\ln 2}{5730}$. From our boundary condition, $0.3 = e^{-kt}$,and solving for t gives us $t = \frac{-\ln(0.3)}{k}$

$$= \ln(\tfrac{10}{3}) \cdot \frac{\tau}{\ln 2} = \left(\frac{\ln(\frac{10}{3})}{\ln 2}\right)\tau \approx 9953 \text{ yr}.$$

13 (b). From (a) $t = \frac{\ln(\frac{10}{3})}{\ln 2}\tau$,and so $\frac{\ln(\frac{10}{3})}{\ln 2}(\tau - 30) \le t \le \frac{\ln(\frac{10}{3})}{\ln 2}(\tau + 30)$

or $9901 \le t \le 10005$ yrs.

13 (c). $\frac{Q(60{,}000)}{Q(0)} = e^{-60{,}000k} = e^{-60{,}000\left(\frac{\ln 2}{5730}\right)} \approx 7.04\left(10^{-4}\right)$.

14. $Q' = -kQ + M$. $Q' + kQ = M$. $p(t) = k$ and $P(t) = kt \Rightarrow e^{kt}Q' + ke^{kt}Q = (e^{kt}Q)' = e^{kt}M$

$\Rightarrow e^{kt}Q = e^{kt}\frac{M}{k} + C \Rightarrow Q = \frac{M}{k} + Ce^{-kt}$. $Q_0 = \frac{M}{k} + C$

$\Rightarrow Q(t) = \frac{M}{k} + \left(Q_0 - \frac{M}{k}\right)e^{-kt} = 50e^{-kt} + \frac{M}{k}\left(1 - e^{-kt}\right)$. $k = \frac{\ln 2}{\tau} = \frac{\ln 2}{3} \approx 0.231 \Rightarrow$

$100 = 50e^{-2k} + \frac{M}{k}\left(1 - e^{-2k}\right) = 31.5 + \frac{M}{.231}(0.37) \Rightarrow M = 42.78$ (mg/yr.).

15. $\tau = \frac{\ln 2}{k} = 8$ days. Substitution gives us $Q(t) = Q_0e^{-kt} = Q_0e^{-\ln 2\frac{t}{\tau}}$ and

$30 = Q_0e^{-\frac{3}{8}\ln 2}$.Finally, $Q_0 = 30e^{\frac{3}{8}\ln 2} \approx 38.9\mu$g

16. $0.99Q_0 = Q_0e^{-kt} \Rightarrow t = \frac{1}{k}\ln\left(\frac{100}{99}\right) = \frac{\tau}{\ln 2}\ln\left(\frac{100}{99}\right) = 4\cdot 10^9 \cdot 0.0145 \approx 0.058\cdot 10^9$=58 million years.

Section 2.5

1 (a). Solving for y', we have $y' = \frac{1}{3}(1 - 2t\cos y)$. Thus, $f(t,y) = \frac{1}{3}(1 - 2t\cos y)$.

1 (b). $\frac{\partial f}{\partial y} = \frac{1}{3}(0 + 2t\sin y) = \frac{2}{3}t\sin y$. f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

1 (c). The largest open rectangle is the entire ty plane, since f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

2 (a). $f(t,y) = \frac{1}{3t}(1 - 2\cos y)$.

2 (b). $\frac{\partial f}{\partial y} = \frac{2}{3t}\sin y$. f and $\frac{\partial f}{\partial y}$ are continuous when $t < 0,\ t > 0$.

2 (c). $R = \left\{(t,y) : t > 0, -\infty < y < \infty\right\}$.

3 (a). Solving for y', we have $y' = -\frac{2t}{1 + y^2}$. Thus, $f(t,y) = -\frac{2t}{1 + y^2}$.

3 (b). $\frac{\partial f}{\partial y} = (-2t)(-1)(1 + y^2)^{-2}(2y) = \frac{4ty}{(1 + y^2)^2}$. f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

3 (c). The largest open rectangle is the entire ty plane, since f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

4 (a). $f(t,y) = \frac{-2t}{1+y^3}$.

4 (b). $\frac{\partial f}{\partial y} = \frac{6ty^2}{(1+y^3)^2}$. f and $\frac{\partial f}{\partial y}$ are continuous everywhere in the ty-plane except on the line $y = -1$.

4 (c). $R = \{(t,y): -\infty < t < \infty, y > -1\}$.

5 (a). Solving for y', we have $y' = \tan t - ty^{\frac{1}{3}}$. Thus, $f(t,y) = \tan t - ty^{\frac{1}{3}}$.

5 (b). $\frac{\partial f}{\partial y} = -\frac{1}{3}ty^{-\frac{2}{3}}$. f and $\frac{\partial f}{\partial y}$ are continuous except on the lines $t = \left(n + \frac{1}{2}\right)\pi$ (where n is an integer) and $y = 0$.

5 (c). The largest open rectangle is $R = \left\{(t,y): -\frac{\pi}{2} < t < \frac{\pi}{2}, 0 < y < \infty\right\}$.

6 (a). $f(t,y) = \frac{t^2 - e^{-y}}{y^2 - 9}$.

6 (b). $\frac{\partial f}{\partial y} = \frac{(y^2 + 2y - 9)e^{-y} - 2t^2 y}{(y^2 - 9)^2}$. f and $\frac{\partial f}{\partial y}$ are continuous everywhere in the ty-plane except $y = \pm 3$.

6 (c). $R = \{(t,y): -\infty < t < \infty, -3 < y < 3\}$.

7 (a). Solving for y', we have $y' = \frac{2 + \tan t}{\cos y}$. Thus, $f(t,y) = \frac{2 + \tan t}{\cos y}$.

7 (b). $\frac{\partial f}{\partial y} = (2 + \tan t)(-1)(\cos y)^{-2}(-\sin y) = (2 + \tan t)\sec y \tan y$. f and $\frac{\partial f}{\partial y}$ are continuous except on the lines $t = \left(n + \frac{1}{2}\right)\pi$ (where n is an integer) and $y = \left(m + \frac{1}{2}\right)\pi$ (where m is an integer).

7 (c). The largest open rectangle is $R = \left\{(t,y): -\frac{\pi}{2} < t < \frac{\pi}{2}, -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$.

8 (a). $f(t,y) = \frac{2 + \tan y}{\cos 2t}$.

8 (b). $\frac{\partial f}{\partial y} = \frac{\sec^2 y}{\cos 2t}$. f and $\frac{\partial f}{\partial y}$ are continuous except where $\tan y$ is not defined and $\cos 2t = 0$, or where $y = \left(n + \frac{1}{2}\right)\pi$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$, and $t = \left(m + \frac{1}{2}\right)\frac{\pi}{2}$, $m = \ldots, -2, -1, 0, 1, 2, \ldots$

8 (c). $R=\left\{(t,y):\frac{3\pi}{4}<t<\frac{5\pi}{4},-\frac{\pi}{2}<y<\frac{\pi}{2}\right\}$.

9 (a). $f(t,y)=\frac{y^2}{t^2}$, $\frac{\partial f}{\partial y}=\frac{2y}{t^2}$. f and $\frac{\partial f}{\partial y}$ are continuous except where $t=0$.

$R=\left\{(t,y):0<t<\infty,-\infty<y<\infty\right\}$.

9 (b). There is no contradiction. If the hypotheses are not satisfied, "bad things need not happen".

10. $\bar{y}(t)=(4+(t-t_0))^{\frac{3}{2}}$, so $\bar{y}(0)=(4-t_0)^{\frac{3}{2}}=1\Rightarrow t_0=3$.

11. $\bar{y}(t)=\frac{2}{\sqrt{1-(t-1)}}$, so $\bar{y}(0)=\frac{2}{\sqrt{2}}=\sqrt{2}$.

12 (a). $z_1(t)=y(t+2)$, so $z_1(-5)=y(-3)=2$.

12 (b). $z_2(t)=y(t-2)$, so $z_2(3)=y(1)=0$.

13 (a). (i) $y'=y(2-y)\Rightarrow y'-2y=-y^2\Rightarrow 1-n=-1=m,\ v=y^{-1}\Rightarrow y=v^{-1}$, thus $y'=-v^2v'$ and $-v^{-2}v'=2v^{-1}-v^{-2}$ or $v'+2v=1,\ v(0)=1$.

(ii) $\left(e^{2t}v\right)'=e^{2t}\Rightarrow e^{2t}v=\frac{1}{2}e^{2t}+C$ or $v=\frac{1}{2}+Ce^{-2t}$. From the initial condition,

$\frac{1}{2}+C=1\Rightarrow C=\frac{1}{2}$, and so $v=\frac{1}{2}\left(1+e^{-2t}\right)$.

(iii) $y=v^{-1}=\frac{2}{1+e^{-2t}}$.

13 (b). $-\infty<t<\infty$

14 (a). (i) $y'=2ty-2ty^2\Rightarrow 1-2=-1=m,\ v=y^{-1}\Rightarrow y=v^{-1}$, thus $-v^{-2}v'=2tv^{-1}-2tv^{-2}$ or $v'+2tv=2t,\ v(0)=-1$.

(ii) $\left(e^{t^2}v\right)'=2te^{t^2}\Rightarrow e^{t^2}v=e^{t^2}+C$ or $v=1+Ce^{-t^2}$. From the initial condition,

$1+C=-1\Rightarrow C=-2$, and so $v=1-2e^{-t^2}$.

(iii) $y=v^{-1}=\frac{1}{1-2e^{-t^2}}$.

14 (b). $-\sqrt{\ln 2}<t<\sqrt{\ln 2}$

15 (a). (i) $m=1-n=-1,\ v=y^{-1}\Rightarrow y=v^{-1}$, thus $y'=-v^{-2}v'=-v^{-1}+e^tv^{-2}\Rightarrow v'=v-e^t$ or $v'-v=-e^t,\ v(-1)=-1$.

(ii) $\left(e^{-t}v\right)'=-1\Rightarrow e^{-t}v=-t+C$ or $v=-te^t+Ce^t$. From the initial condition,

$e^{-1}+Ce^{-1}=-1\Rightarrow C=-(1+e)$, and so $v=e^t(-t-1-e)=-(t+1)e^t-e^{t+1}$.

(iii) $y = v^{-1} = \dfrac{-1}{(t+1)e^{t} + e^{t+1}}$.

15 (b). $-(1+e) < t < \infty$

16 (a). (i) $1-n = 2 = m,\ v = y^{2} \Rightarrow y = v^{\frac{1}{2}}$, thus $y' = \dfrac{1}{2}v^{-\frac{1}{2}}v'$ and $\dfrac{1}{2}v^{-\frac{1}{2}}v' = v^{\frac{1}{2}} + v^{-\frac{1}{2}}$

or $v' = 2v + 2,\ v(0) = 1$.

(ii) $\left(e^{-2t}v\right)' = 2e^{-2t} \Rightarrow e^{-2t}v = -e^{-2t} + C$ or $v = -1 + Ce^{2t}$. From the initial condition,

$-1 + C = 1 \Rightarrow C = 2$, and so $v = -1 + 2e^{2t}$.

(iii) $y = -\sqrt{-1 + 2e^{2t}}$.

16 (b). $-\dfrac{1}{2}\ln 2 < t < \infty$

17 (a). (i) $m = 1-n = 3,\ v = y^{3} \Rightarrow y = v^{\frac{1}{3}}$, thus $y' = \dfrac{1}{3}v^{-\frac{2}{3}}v'$. Then $t \cdot \dfrac{1}{3}v^{-\frac{2}{3}}v' + v^{\frac{1}{3}} = t^{3}v^{-\frac{2}{3}}$, and so

$tv' + 3v = 3t^{3},\ v(1) = 1$.

(ii) $\left(t^{3}v\right)' = 3t^{5} \Rightarrow t^{3}v = \dfrac{t^{6}}{2} + C$ or $v = \dfrac{1}{2}t^{3} + \dfrac{C}{t^{3}}$. From the initial condition, $\dfrac{1}{2} + C = 1 \Rightarrow C = \dfrac{1}{2}$,

and so $v = \dfrac{1}{2}\left(t^{3} + t^{-3}\right)$.

(iii) $y = v^{\frac{1}{3}} = \left(\dfrac{1}{2}\left(t^{3} + t^{-3}\right)\right)^{\frac{1}{3}}$.

17 (b). $0 < t < \infty$

18 (a). (i) $m = 1-n = \dfrac{2}{3},\ v = y^{\frac{2}{3}} \Rightarrow y = v^{\frac{3}{2}}$, thus $y' = \dfrac{3}{2}v^{\frac{1}{2}}v'$. Then $\dfrac{3}{2}v^{\frac{1}{2}}v' - v^{\frac{3}{2}} = tv^{\frac{1}{2}}$, and

so $v' - \dfrac{2}{3}v = \dfrac{2}{3}t,\ v(0) = 4$.

(ii) $\left(e^{-\frac{2}{3}t}v\right)' = \dfrac{2}{3}te^{-\frac{2}{3}t} \Rightarrow e^{-\frac{2}{3}t}v = \dfrac{2}{3}\left(-\dfrac{3}{2}te^{-\frac{2}{3}t} - \dfrac{9}{4}e^{-\frac{2}{3}t}\right) + C$ or $v = -t - \dfrac{3}{2} + Ce^{\frac{2}{3}t}$. From the initial

condition, $-\dfrac{3}{2} + C = 4 \Rightarrow C = \dfrac{11}{2}$, and so $v = -\left(t + \dfrac{3}{2}\right) + \dfrac{11}{2}e^{\frac{2}{3}t}$.

(iii) $y = -\left(\dfrac{11}{2}e^{\frac{2}{3}t} - \left(t + \dfrac{3}{2}\right)\right)^{\frac{3}{2}}$.

18 (b). $-\infty < t < \infty$

19. First, let $z = y + 1,\ z' = -z + tz^{-2},\ 1 - n = 3$.

Therefore, $v = z^3,\ z = v^{\frac{1}{3}},\ z' = \frac{1}{3}v^{-\frac{2}{3}}v' \Rightarrow \frac{1}{3}v^{-\frac{2}{3}}v' + v^{\frac{1}{3}} = tv^{-\frac{2}{3}}$. Then,

$v' + 3v = 3t,\ y(0) = 1 \Rightarrow u(0) = 2 \Rightarrow v(0) = 8$ and

$v = Ce^{-3t} + at + b,\ a + 3(at + b) = 3t \Rightarrow a = 1,\ b = -\frac{1}{3}.$

Therefore, $v = Ce^{-3t} + t - \frac{1}{3},\ v(0) = C - \frac{1}{3} = 8 \Rightarrow C = \frac{25}{3}$.

Then, $v = \frac{25}{3}e^{-3t} + t - \frac{1}{3},\ y = u - 1 = v^{\frac{1}{3}} - 1 = \left(\frac{25}{3}e^{-3t} + t - \frac{1}{3}\right)^{\frac{1}{3}} - 1,\ -\infty < t < \infty$.

20. $y_0 = 3$ by substitution. Differentiating yields

$$y' = \frac{-3e^{-t}}{1-3t} + 3e^{-t}\left(\frac{-1}{(1-3t)^2}\right)(-3) = -\frac{3}{(1-3t)e^t} + e^t\left(\frac{9}{(1-3t)^2e^{2t}}\right) = -y + e^t y^2.$$

Thus $q(t) = e^t$.

Section 2.6

1 (a). Antidifferentiation gives us $\frac{y^2}{2} + \cos t = C$. From the initial condition, we have

$\frac{(-2)^2}{2} + \cos\frac{\pi}{2} = C = 2$. Then we have $y^2 = 4 - 2\cos t,\ y = -\sqrt{4 - 2\cos t}$.

1 (b). $-\infty < t < \infty$

2 (a). $y^2y' = 1$, so $\frac{y^3}{3} - t = C$. From the initial condition, we have $\frac{8}{3} - 1 = \frac{5}{3} = C$. Then we have

$y^3 = 3t + 5 \Rightarrow y = (3t + 5)^{\frac{1}{3}}$.

2 (b). $-\infty < t < \infty$

3 (a). $(y + 1)y' + 1 = 0$, so $\frac{y^2}{2} + y + t = C$. From the initial condition, we have . Then we have

$\frac{y^2}{2} + y + t = 1 \Rightarrow y^2 + 2y + 2(t - 1) = 0,\ y = \frac{-2 \pm \sqrt{4 - 8(t-1)}}{2}$. Since $y(1) = 0$, we only want the

plus sign. Finally, $y = \frac{-2 + \sqrt{4 - 8(t-1)}}{2} = -1 + \sqrt{3 - 2t}$.

3 (b). $-\infty < t \le \frac{3}{2}$

4 (a). $y^{-2}y' - 2t = 0$, so $-y^{-1} - t^2 = C$. From the initial condition, we have $1 - 0 = C$. Then we have

$-y^{-1} = t^2 + 1 \Rightarrow y = \dfrac{-1}{1+t^2}$.

4 (b). $-\infty < t < \infty$

5 (a). $y^{-3}y' - t = 0$, so $\dfrac{y^{-2}}{-2} - \dfrac{t^2}{2} = C$. From the initial condition, we have $C = -\dfrac{1}{8}$. Then we have

$y^{-2} + t^2 = \dfrac{1}{4}$, $y = \dfrac{1}{\sqrt{1/4 - t^2}} = \dfrac{2}{\sqrt{1-4t^2}}$.

5 (b). $-\dfrac{1}{2} < t < \dfrac{1}{2}$

6 (a). $e^{-y}y' + (t - \sin t) = 0$, so $-e^{-y} + \left(\frac{t^2}{2} + \cos t\right) = C$. From the initial condition, we have

$-1 + 1 = 0 = C$. Then we have $e^{-y} = \frac{t^2}{2} + \cos t \Rightarrow y = -\ln\left(\frac{t^2}{2} + \cos t\right)$.

6 (b). $-\infty < t < \infty$

7 (a). $\dfrac{1}{1+y^2}y' - 1 = 0$, so $\tan^{-1}y - t = C$. From the initial condition, we have $C = -\dfrac{\pi}{2}$. Then we

have $\tan^{-1} y = t - \dfrac{\pi}{2}$, $y = \tan\left(t - \dfrac{\pi}{2}\right)$.

7 (b). $0 < t < \pi$

8 (a). $(\cos y)y' + t^{-2} = 0$, so $\sin y - t^{-1} = C$. From the initial condition, we have $0 - (-1) = 1 = C$. Then

we have $\sin y = 1 + t^{-1} \Rightarrow y = \sin^{-1}\left(1 + t^{-1}\right)$.

8 (b). $-\infty < t < -\dfrac{1}{2}$

9 (a). $\dfrac{1}{1-y^2}y' - t = 0$.

By partial fractions, $\dfrac{1}{1-y^2} = \dfrac{-1}{y^2-1} = \dfrac{-1}{(y-1)(y+1)} = \dfrac{-\frac{1}{2}}{y-1} + \dfrac{\frac{1}{2}}{y+1}$, and so $\dfrac{1}{2}\ln\left|\dfrac{y+1}{y-1}\right| - \dfrac{t^2}{2} = C$.

From the initial condition, we have $\dfrac{1}{2}\ln 3 = C$. Then we have

$\ln\left|\dfrac{y+1}{y-1}\right| - t^2 = \ln 3 \Rightarrow \ln\left|\dfrac{1}{3}\left(\dfrac{y+1}{y-1}\right)\right| = t^2$, and solving for y yields $y = \dfrac{3e^{t^2} - 1}{3e^{t^2} + 1}$.

9 (b). $-\infty < t < \infty$

10 (a). $3y^2y' + 2t - 1 = 0$, so $y^3 + t^2 - t = C$. From the initial condition, we have $-1 + 1 - (-1) = 1 = C$. Then we have $y^3 = 1 + t - t^2 \Rightarrow y = \left(1 + t - t^2\right)^{\frac{1}{3}}$.

10 (b). $-\infty < t < \infty$

11 (a). $e^y y' - e^t = 0$, so $e^y - e^t = C$. From the initial condition, we have $C = e - 1$. Then we have $e^y - e^t = e - 1$, $y = \ln\left(e^t + e - 1\right)$.

11 (b). $-\infty < t < \infty$

12 (a). $yy' - t = 0$, so $\dfrac{y^2}{2} - \dfrac{t^2}{2} = C$. From the initial condition, we have $2 - 0 = C$. Then we have $\dfrac{y^2}{2} - \dfrac{t^2}{2} = 2 \Rightarrow y = -\sqrt{4 + t^2}$.

12 (b). $-\infty < t < \infty$

13 (a). $\sec^2 y\left(y'\right) + e^{-t} = 0$, so $\tan y - e^{-t} = C$. From the initial condition, we have $C = 1 - 1 = 0$. Then we have $\tan y = e^{-t}$, $y = \tan^{-1}\left(e^{-t}\right)$.

13 (b). $-\infty < t < \infty$

14 (a). $(2y - \sin y)\left(y'\right) + (t - \sin t) = 0$, so $y^2 + \cos y + \dfrac{t^2}{2} + \cos t = C$. From the initial condition, we have $0 + 1 + 0 + 1 = 2 = C$. Then we have $y^2 + \cos y = 2 - \dfrac{t^2}{2} - \cos t$. There is no explicit solution.

15 (a). $\left(y + 1\right)e^y y' + \left(t - 2\right) = 0$, so $ye^y + \dfrac{\left(t - 2\right)^2}{2} = C$. From the initial condition, we have $C = 2e^2 + \dfrac{1}{2}$. Then we have $ye^y = 2e^2 + \dfrac{1}{2} - \dfrac{\left(t - 2\right)^2}{2}$. There is no explicit solution.

16 (a). $y \ln y - y + \dfrac{t^2}{2} = t + C$. $e \ln e - e + \dfrac{9}{2} = 3 + C \Rightarrow C = \dfrac{3}{2} \Rightarrow -y + y \ln y + \dfrac{t^2}{2} - t = \dfrac{3}{2}$.

17 (a). $\dfrac{e^y y'}{1 + e^y} = 1$, so $\ln\left(1 + e^y\right) = t + C$. From the initial condition, we have $C = \ln 2 - 2$. Then we have $\ln\left(1 + e^y\right) - t = -2 + \ln 2$. We can simplify this expression by taking the natural exponential of each side: $1 + e^y = e^{t-2}e^{\ln 2} = 2e^{t-2} \Rightarrow e^y = 2e^{t-2} - 1 \Rightarrow y = \ln\left[2e^{t-2} - 1\right]$.

17 (b). $2e^{t-2} - 1 > 0 \Rightarrow t - 2 > \ln\left(\frac{1}{2}\right) \Rightarrow t > 2 + \ln\left(\frac{1}{2}\right) \Rightarrow t > 2 - \ln(2)$

18. $y = (4+t)^{-\frac{1}{2}}$, so $y' = -\dfrac{1}{2}(4+t)^{-\frac{3}{2}} = -\dfrac{1}{2}y^3 \Rightarrow y' + \dfrac{1}{2}y^3 = 0,\ y(0) = 4^{-\frac{1}{2}} = \dfrac{1}{2}$.

Therefore, $\alpha = \dfrac{1}{2},\ n = 3,\ y_0 = \dfrac{1}{2}$.

19. $y = \dfrac{6}{(5+t^4)}$, so $y' = 6(-1)(5+t^4)^{-2}(4t^3) = \dfrac{-24t^3}{(5+t^4)^2} = -24t^3\left(\dfrac{y}{6}\right)^2 = -\dfrac{2}{3}t^3y^2$. Then we have

$y' + \dfrac{2}{3}t^3y^2 = 0$, so $\alpha = \dfrac{2}{3},\ n = 3,\ y_0 = \dfrac{6}{5+1} = 1$.

20. $y^3 + t^2 + \sin y = 4 \Rightarrow 3y^2y' + 2t + (\cos y)y' = 0 \Rightarrow (3y^2 + \cos y)y' + 2t = 0$.

When $t = 2,\ y_0^3 + 4 + \sin y_0 = 4 \Rightarrow y_0^3 + \sin y_0 = 0 \Rightarrow y_0 = 0 \Rightarrow y(2) = 0$.

21. First, $y'e^y + ye^yy' + 2t = \cos t$. Then $(1+y)e^yy' + (2t - \cos t) = 0$. At $t_0 = 0$, we have

$y_0e^{y_0} + 0 = 0$, so $y_0 = 0$, and thus $y(0) = 0$.

22. $y^{-2}y' = 2 \Rightarrow -y^{-1} = 2t + C,\ -y_0^{-1} = C \Rightarrow -y^{-1} = 2t - y_0^{-1} \Rightarrow y^{-1} = y_0^{-1} - 2t \Rightarrow y = \dfrac{1}{y_0^{-1} - 2t}$.

Require $y_0^{-1} - 2(4) = 0 \Rightarrow y_0 = \dfrac{1}{8}$.

23 (b). $\dfrac{y'}{y(2-y)} = 1 \Rightarrow \dfrac{1}{2}\dfrac{y'}{y} + \dfrac{1}{2}\dfrac{y'}{2-y} = 1$. Integration gives us $\dfrac{1}{2}\ln|y| - \dfrac{1}{2}\ln|2-y| = t + C$. From the

boundary condition, we obtain $\dfrac{1}{2}\ln 1 - \dfrac{1}{2}\ln 1 = 2 + C$, and solving for C yields $C = -2$.

Therefore, $\ln|y| - \ln|2-y| = 2t - 4 \Rightarrow \ln\left|\dfrac{y}{2-y}\right| = 2t - 4 \Rightarrow \dfrac{y}{2-y} = e^{2t-4} \Rightarrow y = \dfrac{2e^{2t-4}}{1 + e^{2t-4}}$.

24. $y' = 1 + (y+1)^2$. Let $u = y + 1,\ u' = 1 + u^2,\ \dfrac{1}{(1+u^2)}u' = 1 \Rightarrow \tan^{-1}(u) = t + C$.

Then, $y(0) = 0 \Rightarrow u(0) = 1,\ \dfrac{\pi}{4} = 0 + C \Rightarrow \tan^{-1}(u) = t + \dfrac{\pi}{4} \Rightarrow u = y + 1 = \tan\left(t + \dfrac{\pi}{4}\right)$.

Therefore, $y = \tan\left(t + \dfrac{\pi}{4}\right) - 1,\ -\dfrac{3\pi}{4} < t < \dfrac{\pi}{4}$.

25. $y' = t\left((y+2)^2 + 1\right)$. Letting $u = y + 2$, we have $u' = t(u^2 + 1)$, so $\dfrac{1}{u^2+1}u' = t$.

Then $\tan^{-1} u = \dfrac{t^2}{2} + C$. From the initial condition, we have $y(0) = -3$ and $u(0) = -1$, so

$-\dfrac{\pi}{4} = 0 + C,\ C = -\dfrac{\pi}{4}$, and $\tan^{-1} u = \dfrac{t^2}{2} - \dfrac{\pi}{4}$. In terms of y, this reads $y = -2 + \tan\left(\dfrac{t^2}{2} - \dfrac{\pi}{4}\right)$.

Setting $-\frac{\pi}{2} < \frac{t^2}{2} - \frac{\pi}{4} < \frac{\pi}{2}$ and simplifying, we have

$$-\frac{\pi}{2} < t^2 < \frac{3\pi}{2} \Rightarrow |t| < \sqrt{\frac{3\pi}{2}} \Rightarrow -\sqrt{\frac{3\pi}{2}} < t < \sqrt{\frac{3\pi}{2}}.$$

26. $y' = (y+1)^2 \sin t$. $\frac{y'}{(y+1)^2} = \sin t \Rightarrow \frac{-1}{y+1} = -\cos t + C$.

Then, $y(0) = 0 \Rightarrow -1 = -1 + C \Rightarrow C = 0 \Rightarrow \frac{-1}{y+1} = -\cos t$.

Therefore, $y + 1 = \sec t \Rightarrow y = \sec t - 1$.

27. $Q^{-3}Q' + k = 0$, so $\frac{Q^{-2}}{-2} + kt = C'$ and $Q^{-2} = 2kt - C$. From the implicit initial condition, we have $Q_0^{-2} = -C$, so $Q^{-2} = 2kt + Q_0^{-2}$. Solved for Q, we have $Q(t) = \frac{1}{\sqrt{2kt + Q_0^{-2}}} = \frac{Q_0}{\sqrt{1 + 2kQ_0^2 t}}$.

Thus $\frac{1}{2}Q_0 = \frac{Q_0}{\sqrt{1 + 2kQ_0^2\tau}}$, where τ is the half-life of the reactant. Therefore,

$2 = \sqrt{1 + 2kQ_0^2\tau}$, which, solved for τ, gives $\tau = \frac{3}{2kQ_0^2}$. Thus the half-life depends upon Q_0.

28. $Q' = -kQ^2,\ Q(0) = Q_0;\ Q^{-2}Q' = -k \Rightarrow -Q^{-1} = -kt + C,\ C = -Q_0^{-1}$.

Therefore, $Q^{-1} = kt + Q_0^{-1} \Rightarrow Q = \frac{1}{kt + Q_0^{-1}} = \frac{Q_0}{1 + kQ_0 t},\ Q(10) = 0.4Q_0$.

Then, $0.4Q_0 = \frac{Q_0}{1 + kQ_0(10)} \Rightarrow 0.4 + 4kQ_0 = 1 \Rightarrow kQ_0 = 0.15$ and $Q = \frac{Q_0}{1 + .15t}$.

Set $Q = 0.25Q_0$. Then, $0.25 = \frac{1}{1 + .15t} \Rightarrow t = 20$ min.

29 (a). The equation is nonlinear and separable. $\frac{1}{|y|}y' - 1 = 0$.

29 (b). $|y| = \begin{cases} y, & y \ge 0 \\ -y, & y < 0 \end{cases}$. Thus $\int \frac{dy}{|y|} = \begin{cases} \ln y, & y > 0 \\ -\ln y, & y < 0 \end{cases} \Rightarrow y(t) = \begin{cases} y(0)e^{t}, & y > 0 \\ y(0)e^{-t}, & y < 0 \end{cases}$.

Since $y(0) = 1 > 0$, the solution $y(t) = e^t$ of $y' = |y|,\ y(0) = 1$ will be identical to that of $y' = y,\ y(0) = 1$ as long as $y(t) = e^t \ge 0$. This is true for all t, however, and so the two solution curves agree.

29 (c). If $y(0) = -1 < 0$, then the solution of $y' = |y|,\ y(0) = -1$, is $y(t) = -e^{-t}$, but the solution of $y' = y,\ y(0) = -1$, is $y(t) = -e^{t}$.

30. $y' = -y^2$ is graph c. $y' = y^3$ is graph a. $y' = y(4-y)$ is graph b.

31. $\left(\frac{K}{S}+1\right)S' + \alpha = 0$, so $K\ln S + S + \alpha t = C$. From the initial condition, we have

$K\ln S_0 + S_0 = C$, so $K\ln S + S = -\alpha t + K\ln S_0 + S_0$.

32 (a). $y' = f(\alpha t + \beta y + \gamma), z = \alpha t + \beta y + \gamma, z' = \alpha + \beta y' = \alpha + \beta f(z)$. Therefore, $g(z) = \alpha + \beta f(z)$.

32 (b). $y' = f\left(\frac{y}{t}\right), z = \frac{y}{t}, z' = -\frac{y}{t^2} + \frac{y'}{t} = -\frac{z}{t} + \frac{f(z)}{t}$. Therefore, $tz' = g(z) = -z + f(z)$.

33. First, simplify the equation for y': $y' = \dfrac{y/t - 1}{y/t + 1}$ and let $z = \frac{y}{t}$. Then we

have $z' = \frac{1}{t}y' - \frac{1}{t^2}y \Rightarrow tz' = y' - \frac{y}{t}$. Substitution gives us $tz' = \frac{z-1}{z+1} - z, z(2) = 1$. Thus

$tz' = \frac{-z^2-1}{z+1} \Rightarrow z'\frac{z+1}{z^2+1} = -\frac{1}{t}$. Integration gives us $\frac{1}{2}\ln\left(z^2+1\right) + \tan^{-1}(z) = -\ln|t| + C$, and with

our boundary condition $z(2) = 1$ we substitute, simplify and find $C = \frac{3}{2}\ln 2 + \frac{\pi}{4}$. Since

$$z = \frac{y}{t}, \frac{1}{2}\ln\left(\left(\frac{y}{t}\right)^2 + 1\right) + \tan^{-1}\left(\frac{y}{t}\right) + \ln t = \frac{3}{2}\ln 2 + \frac{\pi}{4}, t > 0$$

34. $y' = \frac{y+t}{y+t+1}, z = y + t + 1, y' = f(z) = \frac{z-1}{z}, \alpha = \beta = \gamma = 1$.

$z' = 1 + \frac{z-1}{z} = \frac{2z-1}{z} \Rightarrow \frac{z}{2z-1}z' = 1$. Therefore, $\frac{z}{2} + \frac{1}{4}\ln|2z-1| = t + C$.

$y(-1) = 0 \Rightarrow z(-1) = 0 \Rightarrow C = 1$ and

$\frac{z}{2} + \frac{1}{4}\ln|2z-1| = t + 1 \Rightarrow \frac{y+t+1}{2} + \frac{1}{4}\ln|2y+2t+1| = t + 1 \Rightarrow y - t - 1 + \frac{1}{2}\ln|2y+2t+1| = 0$.

35. Letting $z = t + y$, we have $z' = 1 + y'$, and so $z' = z^2, z(1) = 3$. Separating the variables, we

have $\frac{z'}{z^2} = 1 \Rightarrow -\frac{1}{z} = t + C$. With our boundary condition, we substitute, simplify, and

find $C = -\frac{4}{3}$. Thus $z = \frac{3}{4-3t}$ and so $y = \frac{3-4t+3t^2}{4-3t}$.

36. $y' = \frac{1}{2t+3y+1}, y(1) = 0, z = 2t + 3y + 1, z' = 2 + \frac{3}{z}, z(1) = 3$. $z' = \frac{2z+3}{z} \Rightarrow \frac{z}{2z+3}z' = 1$.

Integrating, $\frac{z}{2} - \frac{3}{4}\ln|2z+3| = t + C, \frac{3}{2} - \frac{3}{4}\ln 9 = 1 + C \Rightarrow C = \frac{1}{2} - \frac{3}{4}\ln 9$.

Therefore, $\frac{z}{2}-\frac{3}{4}\ln|2z+3| = t+\frac{1}{2}-\frac{3}{4}\ln 9 \Rightarrow z-\frac{3}{2}\ln|2z+3| = 2t+1-\frac{3}{2}\ln 9$ and

$2t+3y+1-\frac{3}{2}\ln\left|\frac{4t+6y+5}{9}\right| = 2t+1 \Rightarrow y-\frac{1}{2}\ln\left|\frac{4t+6y+5}{9}\right| = 0$.

37. Letting $z = 2t+y, z' = 2+y'$ and we have $z'-2 = z+\frac{1}{z}, z(1) = 3$. Let us rewrite this as $z'\frac{z}{z^2+2z+1} = 1$. To integrate, we write $\frac{z}{z^2+2z+1}z' = \left(\frac{1}{z+1}-\frac{1}{(z+1)^2}\right)z' = 1$. Integrating, $\ln|z+1|+\frac{1}{z+1} = t+C$. With our boundary condition, we substitute, simplify, and find $C = \ln 4-\frac{3}{4}$. Therefore, $\ln|z+1|+\frac{1}{z+1} = t+\ln 4-\frac{3}{4} \Rightarrow \ln|2t+y+1|+\frac{1}{2t+y+1} = t+\ln 4-\frac{3}{4}$.

38. $t^2y' = y^2-ty, y(-2) = 2, y' = \left(\frac{y}{2}\right)^2-\frac{y}{t}, z = \frac{y}{t}, z' = \frac{1}{t}(z^2-z)-\frac{z}{t} \Rightarrow \frac{1}{z(z-2)}z' = \frac{1}{t}, z(-2) = -1$.

Integrating, $-\frac{1}{2}\ln|z|+\frac{1}{2}\ln|z-2| = \ln|t|+C, C = \frac{1}{2}\ln\left(\frac{3}{4}\right)$. Therefore,

$$\frac{1}{2}\ln\left|\frac{z-2}{t^2z}\right| = \frac{1}{2}\ln\left(\frac{3}{4}\right) \Rightarrow \left|\frac{z-2}{t^2z}\right| = \frac{3}{4} \Rightarrow \left|\frac{\frac{y}{t}-2}{yt}\right| = \frac{3}{4}.$$

When $t = -2, y = 2$, then $\frac{y}{t}-2 = -3$ and $yt = -4$, so $\frac{y}{t}-2 = \frac{3}{4}yt$.

Solving for y, $y = \frac{8t}{4-3t^2}, t < -\frac{\sqrt{3}}{2}$.

39 (c). $\sqrt{1-y^2} = 0 \Rightarrow y_e = \pm 1$. If $y(t^*) = y_e, y(t) = y_e$ for $t > t^*$

Section 2.7

1. $H_y = N = 2y-t$, and integration gives us $H = y^2-ty+\phi(t)$. Differentiation yields $H_t = M = -y+\phi'(y).\phi'(t) = 2t$, and integration gives us $\phi(t) = t^2$.
Thus $H = y^2-ty+t^2 = C$. Substitution with the boundary condition gives us $H(1,0) = 1 = C$, and so $y^2-ty+t^2-1 = 0$. Solving for y: $y = \frac{t\pm\sqrt{4-3t^2}}{2}$. Since $y(1) = 0$, $y = \frac{t-\sqrt{4-3t^2}}{2}$.

2. $M = y + t^3,\ N = t + y^3,\ M_y = N_t = 1$, so the equation is exact.

$H_t = M = y + t^3 \Rightarrow H = yt + \dfrac{t^4}{4} + \phi(y)$ and

$H_y = t + \phi'(y) = N = t + y^3 \Rightarrow \phi'(y) = y^3 \Rightarrow \phi(y) = \dfrac{y^4}{4}$.

Therefore, $yt + \dfrac{t^4}{4} + \dfrac{y^4}{4} = C,\ y(0) = -2 \Rightarrow C = 4$ and $\dfrac{y^4}{4} + yt + \dfrac{t^4}{4} = 4 \Rightarrow y^4 + 4yt + t^4 = 16$.

3. First, rewrite the given equation as $\dfrac{1}{y^2+1}y' - \left(3t^2+1\right) = 0,\ y(0) = 1$. Then $H_t = M = -3t^2 - 1$, and integration gives us $H = -t^3 - t + \phi(y)$. Differentiation yields $H_y = N = \dfrac{1}{y^2+1}.\phi'(y) = \dfrac{1}{y^2+1}$, and integration gives us $\phi(y) = \tan^{-1} y$.

Thus $H = -t^3 - t + \tan^{-1} y$. Substitution with the boundary condition gives us $H(0,1) = \dfrac{\pi}{4}$, and so $-t^3 - t + \tan^{-1} y = \dfrac{\pi}{4}$. The explicit solution is $y = \tan\left(t^3 + t + \dfrac{\pi}{4}\right)$.

4. $H_y = N = y^3 + \cos t$, and integration gives us $H = \dfrac{y^4}{4} + y\cos t + \phi(t)$. Differentiation yields $H_t = M = -y\sin t + \phi'(t).\phi'(t) = -2$, and integration gives us $\phi(t) = -2t$. Thus $H = \dfrac{y^4}{4} + y\cos t - 2t = C$. Substitution with the boundary condition gives us $H(0,-1) = -\dfrac{3}{4} = C$, and so $y^4 + 4y\cos t - 8t + 3 = 0$.

5. $H_t = M = e^t e^y + 3t^2$, and integration gives us $H = e^t e^y + t^3 + \phi(y)$. Differentiation yields $H_y = N = e^t e^y + \phi'(y).\phi'(y) = 2y$, and integration gives us $\phi(y) = y^2$. Thus $H = e^t e^y + t^3 + y^2$. Substitution with the boundary condition gives us $H(0,0) = 1$, and so $e^t e^y + t^3 + y^2 = 1$.

6. $H_y = N = y^3 - t^3$, and integration gives us $H = \dfrac{y^4}{4} - yt^3 + \phi(t)$. Differentiation yields $H_t = M = -3yt^2 + \phi'(t).\phi'(t) = -1$, and integration gives us $\phi(t) = -t$. Thus $H = \dfrac{y^4}{4} - yt^3 - t = C$. Substitution with the boundary condition gives us $H(-2,-1) = -\dfrac{23}{4} = C$, and so $y^4 - 4yt^3 - 4t + 23 = 0$.

7. $H_t = M = ty^2 + \cos t$, and integration gives us $H = \dfrac{1}{2}t^2y^2 + \sin t + \phi(y).H_y = t^2y + \phi'(t) = N = e^{2y} + t^2y.\ \ \phi'(y) = e^{2y}$. Thus $H = \dfrac{1}{2}t^2y^2 + \sin t + \dfrac{1}{2}e^{2y}$. Substitution with the boundary condition gives us $H\left(\dfrac{\pi}{2},0\right) = 1 + \dfrac{1}{2} = C$, and so $\dfrac{1}{2}t^2y^2 + \sin t + \dfrac{1}{2}e^{2y} = \dfrac{3}{2}$.

8. $M = y\cos(ty) + 1,\ N = t\cos(ty) + 2ye^{y^2},\ M_y = N_t = \cos(ty) - ty\sin(ty)$, so the equation is exact. $H_t = M = y\cos(ty) + 1 \Rightarrow H = \sin(ty) + t + \phi(y)$ and $H_y = t\cos(ty) + \phi'(y) = N = t\cos(ty) + 2ye^{y^2} \Rightarrow \phi'(y) = 2ye^{y^2}$, and so $\phi = e^{y^2}$. From the initial condition, we have $0 + \pi + 1 = C$, and thus $\sin(ty) + t + e^{y^2} = \pi + 1$.

9. $N_t = 2y,\ M_y = 2y$. $H_t = M = y^2 - 1$, and integration gives us $H = \left(y^2 - 1\right)t + \phi(y)$. $H_y = N = 2ty + \dfrac{1}{y}$. Thus $H = \left(y^2 - 1\right)t + \ln|y|$. Substitution with the boundary condition gives us $H(1{,}1) = 0$, and so $\left(y^2 - 1\right)t + \ln|y| = 0$.

10. $H_y = N = 2y\ln t - t\sin y$, and integration gives us $H = y^2\ln t + t\cos y + \phi(t)$. Differentiation yields $H_t = M = \dfrac{y^2}{t} + \cos y + \phi'(t)$. $\phi'(t) = 0$. Thus $H = y^2\ln t + t\cos y = C$. Substitution with the boundary condition gives us $H(2{,}0) = 2 = C$, and so $y^2\ln t + t\cos y - 2 = 0$.

11. $N_t = 2 = M_y$. Integration gives us $M(t,y) = 2y + \phi(t)$.

12. $N = t^2 + y^2\sin t,\ N_t = M_y = 2t + y^2\cos t$. Thus $M = 2ty + \dfrac{y^3}{3}\cos t + \phi(t)$.

13. $N_t = e^y + 1 = M_y$. Integration gives us $M(t,y) = e^y + y + \phi(t)$.

14. $M_y = 1 = N_t$. Integration gives us $N(t,y) = t + \phi(y)$.

15. $M_y = 2y\sin t = N_t$. Integration gives us $N(t,y) = -2y\cos t + \phi(y)$.

16. $M_y = 2ye^{y^2} + 2t = N_t$. Integration gives us $N(t,y) = 2tye^{y^2} + t^2 + \phi(y)$.

17. From the boundary condition, we set t=0 and obtain $1 + y_0^2 = 5$ from the implicit solution. Thus $y_0 = \pm 2$, and $M = H_t = 3t^2y + e^t$ and $N = H_y = t^3 + 2y$.

18. $2ty + \cos(ty) + y^2 = 2,\quad y(0) = y_0$. $N = H_y = 2t - t\sin(ty) + 2y,\quad M = H_t = 2y - y\sin(ty)$.
$0 + 1 + y_0^2 = 2 \Rightarrow y_0 = \pm 1$

19. From the boundary condition, we set t=0 and obtain $\ln(y_0) + 1 = 1$. Thus $y_0 = 1$, and so $M = H_t = \dfrac{2}{2t + y} + 2t + ye^{yt}$ and $N = H_y = \dfrac{1}{2t + y} + te^{yt}$.

20. $y^3 + 4ty + t^4 + 1 = 0,\quad y(0) = y_0$. $N = H_y = 3y^2 + 4t,\quad M = H_t = 4y + 4t^3$.
$y_0^3 + 1 = 0 \Rightarrow y_0 = -1$

23 (b). Multiplying by μ, we have $4t^{1/2}yy' + \left(y^2 - t\right)t^{-1/2} = 0$, $y(1) = 0$. Extracting M and N and differentiating, we have $M_y = 2yt^{-1/2}$ and $N_t = 2yt^{-1/2}$. Thus the equation is exact.

23 (c). $H_t = M = y^2t^{-1/2} - t^{-1/2}$. Integration gives us $H = 2y^2t^{1/2} - \dfrac{2}{3}t^{3/2} + \phi(y)$. Differentiation gives us $H_y = N = 4t^{1/2}y + \phi'(y)$, and thus $\phi'(y) = 0$. Our implicit solution is thus $2y^2t^{1/2} - \dfrac{2}{3}t^{3/2} = C$. From the boundary condition, we have $y(1) = 1$, and substitution yields $C = \dfrac{4}{3}$.

Solving the resulting equation for y yields $y = \pm\sqrt{\tfrac{1}{3}\left(t + 2t^{-1/2}\right)}$. We choose only the positive root here after examining our boundary condition.

24 (b). Multiplying by μ, we have $\left(t^2y + y^{-1}\right)y' + ty^2 = 0$, $y(0) = 1$.

24 (c). $H_t = M = ty^2$. Integration gives us $H = \dfrac{t^2y^2}{2} + \phi(y)$. Differentiation gives us $H_y = N = t^2y + \phi'(y)$, and thus $\phi'(y) = y^{-1}$. Our implicit solution is thus $\dfrac{t^2y^2}{2} + \ln|y| = C$. From the boundary condition, we have $y(0) = 1$, and substitution yields $C = 0$. The solution is: $t^2y^2 + 2\ln|y| = 0$

25 (b). $N_t = 2$, $M_y = 1$. $\dfrac{N_t - M_y}{M} = \dfrac{1}{y}$, and thus $\mu = e^{\ln y} = y$. Our new problem is thus $\left(2ty + y^2\right)y' + y^2 = 0$, $y(2) = -3$.

25 (c). $H_t = M = y^2$ and integration gives us $H = y^2t + \phi(y)$. Differentiation gives us $H_y = N = 2yt + \phi'(y)$. Further differentiation shows us the equation is exact. Integration gives us $\phi(y) = \dfrac{1}{3}y^3$. Thus $H = y^2t + \dfrac{1}{3}y^3$, and substitution with our boundary condition gives us our explicit solution: $H(2,-3) = y^2t + \dfrac{1}{3}y^3 = 9$, and we can simplify this to $3y^2t + y^3 = 27$.

26 (b). $\dfrac{M_y - N_t}{N} = \dfrac{5}{t}$, and thus $\mu = e^{5\ln t} = t^5$. Our new problem is thus $t^6y^2y' + 2t^5y^3 - t^5 = 0$, $y(1) = -1$.

26 (c). $H_t = M = 2t^5y^3 - t^5$ and integration gives us $H = \dfrac{1}{3}y^3t^6 - \dfrac{t^6}{6} + \phi(y)$. Differentiation gives us $H_y = N = y^2t^6 + \phi'(y)$. $\phi'(y) = 0$. Thus $H = \dfrac{1}{3}y^3t^6 - \dfrac{t^6}{6}$, and substitution with our boundary condition gives us our explicit solution: $H(1,-1) = C = -\dfrac{1}{2}$, and we can simplify this in its general form to $y = \sqrt[3]{\dfrac{t^6 - 3}{2t^6}}$.

27 (b). $\dfrac{M_y - N_t}{N} = \dfrac{2y - y}{ty} = \dfrac{1}{t}$, and thus $\mu = e^{P(t)} = e^{\ln t} = t$. Our new problem is thus $t^2yy' + ty^2 + te^t = 0$, $y(1) = -2$.

27 (c). $H_t = M = ty^2 + te^t$ and integration gives us $H = \dfrac{1}{2}y^2t^2 + te^t - e^t + \phi(y)$. Differentiation gives us $H_y = N = yt^2 + \phi'(y)$. Further differentiation shows us the equation is exact. Integration gives us $\phi(y) = 0$. Thus $H = \dfrac{1}{2}y^2t^2 + te^t - e^t$, and substitution with our boundary condition gives us our explicit solution: $H(1,-2) = C = 2$, and we can simplify this in its general form to $y = \pm\sqrt{\dfrac{4 + 2e^t - 2te^t}{t^2}}$. We will choose only the negative answer for y here after substituting our boundary condition into the final solution.

28 (b). $\dfrac{N_t - M_y}{M} = \dfrac{1}{y}$, and thus $\mu = y$. Our new problem is thus $\left(3ty^2 + 2y\right)y' + y^3 = 0,\ y(-1) = -1$.

28 (c). $H_t = M = y^3$ and integration gives us $H = y^3t + \phi(y)$. Differentiation gives us $H_y = N = 3y^2t + \phi'(y)$. $\phi'(y) = 2y$. Thus $H = y^3t + y^2$, and substitution with our boundary condition gives us our explicit solution: $H(-1,-1) = C = 2$, and we can simplify this in its general form to $y^3t + y^2 - 2 = 0$.

Section 2.8

1. From equation (5), we see that the solution of $P' = r\left(1 - \dfrac{P}{P_e}\right)P;\ P(0) = P_0$

   is $P(t) = \dfrac{P_0P_e}{P_0 - \left(P_0 - P_e\right)e^{-rt}}$. Assume the following values:

   r=0.1; $P_e = 3$; $P(t)$=0.9P_e=2.7; and $P_0 = 0.1$. We then substitute these values into the solution and solve for t: $t \approx 55.645$ years.

2. $P_0 = 5;\ P(t) = 1.1P_e = 3.3 \Rightarrow \text{t} \approx 14.8$ years

3. Assume $P(3) = 2$. We then substitute into the solution and solve for P_0: $P_0 = \dfrac{6e^{-0.3}}{1 + 2e^{-0.3}} \approx 1.7911$ million.

4 (a). $P^2 - P - M = P^2 - P + \dfrac{3}{16} = 0 \Rightarrow P_e = \dfrac{1}{4}, \dfrac{3}{4}$. $P' > 0$ for $\dfrac{1}{4} < P < \dfrac{3}{4}, P' < 0$ for $0 < \text{P} < \dfrac{1}{4}, P > \dfrac{3}{4}$

4 (b). $\lim\limits_{t\to\infty} P(t) = \dfrac{3}{4}$ since $\text{P}(0) > \dfrac{3}{4}$ (see direction field 11).

5 (a). $P^2 - P - M = P^2 - P + \dfrac{3}{16} = 0$. Solving for P yields $P_e = \dfrac{1 \pm 1/2}{2} = \dfrac{3}{4}, \dfrac{1}{4}$.

   $P' > 0$ for $\dfrac{1}{4} < P < \dfrac{3}{4}, P' < 0$ for $0 < \text{P} < \dfrac{1}{4}, P > \dfrac{3}{4}$

5 (b). $\lim\limits_{t\to\infty} P(t) = \dfrac{3}{4}$ since $\dfrac{1}{4} < \text{P}(0) < \dfrac{3}{4}$ (see direction field 11).

6 (a). $P^2 - P - M = P^2 - P + \dfrac{1}{4} = 0 \Rightarrow P_e = \dfrac{1}{2}$.

   $P' < 0$ for $0 < \text{P} < \dfrac{1}{2}, P > \dfrac{1}{2}$

6 (b). $\lim\limits_{t\to\infty} P(t) = 0$ since $0 < \text{P}(0) < \dfrac{1}{2}$ (see direction field 12).

7 (a). $P^2 - P - M = P^2 - P + \dfrac{1}{4} = 0$. Solving for P yields $P_e = \dfrac{1 \pm \sqrt{1-1}}{2} = \dfrac{1}{2}$.

   $P' < 0$ for $0 < \text{P} < \dfrac{1}{2}, P > \dfrac{1}{2}$

7 (b). $\lim\limits_{t\to\infty} P(t) = \dfrac{1}{2}$ since $\text{P}(0) > \dfrac{1}{2}$ (see direction field 12).

8 (a). $P^2 - P - M = P^2 - P + \frac{1}{4} = 0 \Rightarrow P_e = \frac{1}{2}$.

$P' < 0$ for $0 < \mathrm{P} < \frac{1}{2}, P > \frac{1}{2}$

8 (b). $\lim_{t\to\infty} P(t) = \frac{1}{2}$ since $\mathrm{P}(0) = \mathrm{P_e} = \frac{1}{2}$ (see direction field 12).

9 (a). $P^2 - P - M = P^2 - P - 2 = 0$. Solving for P yields $P = 2, -1$. But $P_e = 2$ is the only possible equilibrium population. $P' > 0$ for $0 < \mathrm{P} < 2, P' < 0$ for $P > 2$

9 (b). $\lim_{t\to\infty} P(t) = 2$ since $\mathrm{P}(0) = 0$ (see direction field 13).

10 (a). $P^2 - P - M = P^2 - P - 2 = 0 \Rightarrow P_e = 2$. $P' > 0$ for $0 < \mathrm{P} < 2, P' < 0$ for $P > 2$

10 (b). $\lim_{t\to\infty} P(t) = 2$ since $\mathrm{P}(0) = 4$ (see direction field 13).

11. The equilibrium solutions are the roots of $P^2 - P_eP - P_eM = 0$. Since the roots are 2 and 1, we know that $P^2 - P_eP - P_eM = (P-2)(P-1) = P^2 - 3P + 2$. Therefore, $P_e = 3$ and $M = -\frac{2}{3}$.

12. $P^2 - P_eP - P_eM = (P-2)^2 = P^2 - 4P + 4$. Therefore, $P_e = 4$ and $M = -1$.

13. The equilibrium solutions are the roots of $P^2 - P_eP - P_eM = 0$. Since the roots are -1 and 2, we know that $P^2 - P_eP - P_eM = (P-2)(P+1) = P^2 - P - 2$. Therefore, $P_e = 1$ and $M = 2$.

14 (a). $P^2 - P - M = (P - P_1)(P - P_2) \Rightarrow -M = P_1P_2 > 0 \Rightarrow M < 0$. Therefore, migration out of the colony.

14 (bc). $P' = -(P-P_1)(P-P_2) \Rightarrow \int \frac{dP}{(P-P_1)(P-P_2)} = -t + C$. Since

$(P_1 - P_2) > 0, \frac{1}{P_1 - P_2}\ln\left|\frac{P-P_1}{P-P_2}\right| = -t + C \Rightarrow \left|\frac{P-P_1}{P-P_2}\right| = Ke^{-\lambda t}$. $K = \left|\frac{P_0 - P_1}{P_0 - P_2}\right|$ and $\lambda = P_1 - P_2 > 0$.

(i). If $P(0) > P_1 > P_2$, then $P(t) - P_1 > 0$ and $P(t) - P_2 > 0$ and $K > 0$. Therefore,

$\frac{P-P_1}{P-P_2} = \left(\frac{P_0-P_1}{P_0-P_2}\right)e^{-\lambda t}$. Solving for P: $P(t) = \dfrac{P_1 - P_2\left(\frac{P_0-P_1}{P_0-P_2}\right)e^{-\lambda t}}{1 - \left(\frac{P_0-P_1}{P_0-P_2}\right)e^{-\lambda t}}$. Since

$0 < \left(\frac{P_0-P_1}{P_0-P_2}\right) < 1$ and $\lambda > 0$, the denominator remains positive for all $t \ge 0$.

In this case, $\lim_{t\to\infty} P(t) = P_1$.

(ii). If $P_1 > P(0) > P_2$, then $P(t) - P_1 < 0$ and $P(t) - P_2 > 0$ and $K < 0$. Therefore,

$-\frac{P-P_1}{P-P_2} = -\left(\frac{P_0-P_1}{P_0-P_2}\right)e^{-\lambda t}$. Solving for P: $P(t) = \dfrac{P_1 - P_2\left(\frac{P_0-P_1}{P_0-P_2}\right)e^{-\lambda t}}{1 - \left(\frac{P_0-P_1}{P_0-P_2}\right)e^{-\lambda t}}$ as in (i). Since

$\left(\frac{P_0-P_1}{P_0-P_2}\right) < 0$ and $\lambda > 0$, the denominator ≥ 1 for all $t \ge 0$. In this case, $\lim_{t\to\infty} P(t) = P_1$.

(iii). If $P_1 > P_2 > P(0)$, then $P(t) - P_1 < 0$ and $P(t) - P_2 < 0$ and $K > 0$. Therefore,

$\dfrac{P - P_1}{P - P_2} = \left(\dfrac{P_0 - P_1}{P_0 - P_2}\right)e^{-\lambda t}$. Solving for P: $P(t) = \dfrac{P_1 - P_2\left(\dfrac{P_0 - P_1}{P_0 - P_2}\right)e^{-\lambda t}}{1 - \left(\dfrac{P_0 - P_1}{P_0 - P_2}\right)e^{-\lambda t}}$ as in (i). Since

$\left(\dfrac{P_0 - P_1}{P_0 - P_2}\right) = \left(\dfrac{P_1 - P_0}{P_2 - P_0}\right) > 1$ and $\lambda > 0$, the numerator becomes zero when $e^{-\lambda t} = \dfrac{P_1}{P_2}\left(\dfrac{P_1 - P_0}{P_2 - P_0}\right)^{-1}$.

Call this time t_1. The denominator becomes zero when $e^{-\lambda t} = \left(\dfrac{P_1 - P_0}{P_2 - P_0}\right)^{-1}$. Call this time t_2.

Note that $t_2 > t_1$. Therefore, $t^* = \lambda^{-1}\ln\left(\dfrac{P_1 - P_0}{P_2 - P_0}\right)$. In this case, $P(t_1) = 0$ and the model ceases to be valid for $t > t_1$.

15 (a). $-(P^2 - P - M) = -(P - P_1)^2 \Rightarrow -M = P_1^2 > 0 \Rightarrow M < 0$. Therefore, migration out of the colony.

15 (b). $-(P^2 - P - M) = -(P - P_1)^2 \Rightarrow 2P_1 = 1 \Rightarrow P_1 = P_2 = \dfrac{1}{2}$. Integrating,

$\displaystyle\int \frac{dP}{(P - P_1)^2} = -\frac{1}{P - P_1} = -t + C$ with $C = -\dfrac{1}{P_0 - P_1}$. Therefore,

$$-\frac{1}{P - P_1} = -t - \frac{1}{P_0 - P_1} \Rightarrow \frac{1}{P - P_1} = t + \frac{1}{P_0 - P_1} \Rightarrow P(t) = P_1 + \frac{1}{t + \dfrac{1}{P_0 - P_1}}.$$

15 (c). If $P_0 > P_1$, then $\lim\limits_{t\to\infty} P(t) = P_1$ since the denominator is non-zero.

If $P_0 < P_1$, then $P(t) = P_1 + \dfrac{1}{t - \dfrac{1}{P_1 - P_0}} = P_1 - \dfrac{1}{\dfrac{1}{P_1 - P_0} - t}$ and $\lim\limits_{t \to \frac{1}{P_1 - P_0}} P(t) = -\infty$. Also,

$P(t) = 0$ when $t - \dfrac{1}{P_1 - P_0} = \dfrac{1}{P_1}$ and the model ceases to be valid.

16. $P' = r(t)\left(1 - \dfrac{P}{P_e}\right)P \Rightarrow \dfrac{P'}{P\left(\dfrac{P}{P_e} - 1\right)} = -r(t)$. Integrating,

$\ln\left|\dfrac{P - P_e}{P}\right| = -R(t) + C \Rightarrow \left|\dfrac{P - P_e}{P}\right| = Ke^{-R(t)}, K = \dfrac{P_0 - P_e}{P_0}$. Therefore,

$$P(t) = \frac{P_e}{1 - Ke^{-R(t)}} = \frac{P_0 P_e}{P_0 - (P_0 - P_e)e^{-R(t)}}.$$

17. The IVP is $P' = (1 + \sin 2\pi t)(1 - P)P$, $P(0) = \dfrac{1}{4}$. We can separate the variables and obtain $\dfrac{P'}{P(1-P)} = \left(\dfrac{1}{P} + \dfrac{1}{1-P}\right)P' = 1 + \sin 2\pi t$. Integrating both sides of this equation gives us $\ln|P| - \ln|1 - P| = \ln\left|\dfrac{P}{1-P}\right| = t - \dfrac{1}{2\pi}\cos 2\pi t + C$. With the initial condition, we can substitute and solve for C: $C = \dfrac{1}{2\pi} - \ln(3)$.

Thus our explicit solution is $\ln\left|\frac{P}{1-P}\right| + \ln 3 = t + \frac{1}{2\pi}(-\cos 2\pi t + 1) \Rightarrow \frac{3P}{1-P} = e^{t+\frac{1}{2\pi}(-\cos 2\pi t+1)}$,

which we can simplify to $P(t) = \frac{1}{3e^{-(t+\frac{1}{2\pi}(-\cos 2\pi t+1))}+1}$. Therefore, $\lim_{t\to\infty} P(t) = 1$.

18. $P' = k(N-P)P \Rightarrow \frac{P'}{P(P-N)} = -k$. Integrating,

$\ln\left|\frac{N-P}{P}\right| = -kNt + NC \Rightarrow \left|\frac{N-P}{P}\right| = Ke^{-kNt}, K = \frac{N-P_0}{P_0}$. Therefore,

$P(t) = \frac{N}{1+Ke^{-KNt}} = \frac{NP_0}{P_0 + (N-P_0)e^{-kNt}}$.

19. The IVP is $P' = k(2-P)P,\ P(0) = 0.1$. We also know that $P(1) = 0.2$. We then separate variables and obtain $\frac{P'}{(2-P)P} = \frac{1}{2}\left(\frac{1}{2-P} + \frac{1}{P}\right)P' = k$. Integrating both sides of this equation gives us $-\frac{1}{2}\ln(2-P) + \frac{1}{2}\ln P = \frac{1}{2}\left[\ln\left(\frac{P}{2-P}\right)\right] = kt + C$. With the initial condition, we can substitute and solve for C: $C = \frac{1}{2}\ln\left(\frac{0.1}{1.9}\right)$. Thus our explicit solution is

$\frac{1}{2}\left[\ln\left(\frac{P}{2-P}\right)\right] = kt + \frac{1}{2}\ln\left(\frac{1}{19}\right)$. Using the second boundary condition and substituting, we have $\frac{1}{2}\ln\left(\frac{0.2}{1.8}\right) = k + \frac{1}{2}\ln\left(\frac{1}{19}\right)$. Solving for k yields $k = \frac{1}{2}\ln\left(\frac{19}{9}\right)$. Our explicit solution is

thus $\frac{1}{2}\left[\ln\left(\frac{P}{2-P}\right)\right] = \frac{1}{2}\ln\left(\frac{19}{9}\right)t + \frac{1}{2}\ln\left(\frac{1}{19}\right)$, which we can simplify to read $P = \frac{\frac{2}{19}\left(\frac{19}{9}\right)^t}{1+\frac{1}{19}\left(\frac{19}{9}\right)^t}$.

At t=5, $P \approx 1.3763$ million infected.

20 (a). $(A-B)' = -kAB + kAB = 0,\ A(t) - B(t) = A(0) - B(0) = 5 - 2 = 3$ moles.

20 (b). $B = A - 3,\ A' = -kA(A-3) = k(3-A)A,\ A(0) = 5$.

20 (c). $A(1) = 4,\ A' = 3k\left(1 - \frac{A}{3}\right)A$. Using equation (5), $A(t) = \frac{5\cdot 3}{5-(5-3)e^{-3kt}}$. Thus $A(t) = \frac{15}{5-2e^{-3kt}}$.

We know that $A(1) = 4$, so $\frac{15}{5-2e^{-3k}} = 4$. Solving for e^{-3k} yields $e^{-3k} = \frac{5}{8}$. Thus

$A(4) = \frac{15}{5-2\left(\frac{5}{8}\right)^4} = 3.195$ moles. $B = A - 3 = 0.195$ moles.

21. $P' = r\left(1 - \frac{P}{P_e}\right)P,\ P(0) = P_0$. Therefore, $P' - rP = -\frac{r}{P_e}P^2, v = P^{-1} \Rightarrow P = \frac{1}{v}$ and $P' = -v^{-2}v'$.

Thus, $-v^{-2}v' - rv^{-1} = -\frac{r}{P_e}v^{-2} \Rightarrow v' + rv = \frac{r}{P_e} \Rightarrow \left(e^{rt}v\right)' = \frac{r}{P_e}e^{rt} \Rightarrow e^{rt}v = \frac{1}{P_e}e^{rt} + C$. Using the boundary condition and substituting,

$v = \frac{1}{P_e} + Ce^{-rt}, v(0) = \frac{1}{P_0} \Rightarrow C = \frac{1}{P_0} - \frac{1}{P_e}$ and $\frac{1}{P} = \frac{1}{P_e} + \left(\frac{1}{P_0} - \frac{1}{P_e}\right)e^{-rt}$. Solving for P,

$P = \frac{P_0 P_e}{P_0 + (P_e - P_0)e^{-rt}}$.

Section 2.9

1. With $v_0 = 0, v = -\frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$. Setting $v = -\frac{1}{2}\frac{mg}{k}$ gives us $1 - e^{-\frac{k}{m}t} = \frac{1}{2}$.

Thus $e^{-\frac{k}{m}t} = \frac{1}{2}, \frac{k}{m}t = \ln 2, t = \frac{m}{k}\ln 2$.

2 (a). $m\frac{dv}{dt} + kv = 0 \Rightarrow v(t) = v_0 e^{-\frac{k}{m}t}, m = \frac{3000}{32}\, slug$

$\frac{v(4)}{v_0} = \frac{50}{220} = e^{-k\cdot\frac{32}{3000}\cdot 4} \Rightarrow \ln\left(\frac{22}{5}\right) = \frac{128}{3000}k$. Then, $k = \frac{3000}{128}\ln\left(\frac{22}{5}\right) = 34.725\, lb\cdot \text{sec}/\, ft$.

2 (b). $d = \int_0^4 v(t)dt = v_0\int_0^4 e^{-\frac{k}{m}t}dt = v_0\left(-\frac{m}{k}e^{-\frac{k}{m}t}\right)\Big|_0^4 = \frac{mv_0}{k}\left(1 - e^{-\frac{4k}{m}}\right)$

$= \frac{3000}{32}\left(220\cdot\frac{5280}{3600}\right)\left(\frac{1}{34.725}\right)\left(\frac{170}{220}\right) \approx 673\, ft$.

3. $mv' + \kappa v^2 = 0 \Rightarrow \frac{v'}{v^2} = -\frac{\kappa}{m} \Rightarrow -v^{-1} = -\frac{\kappa}{m}t + C, C = -v_0^{-1}$. Then we

have $v^{-1} = \frac{\kappa}{m}t + v_0^{-1} \Rightarrow v = \frac{v_0}{1 + \frac{\kappa}{m}v_0 t}$. From the condition provided, we

have $\frac{v(4)}{v_0} = \frac{50}{220} = \frac{1}{1 + 4\frac{\kappa}{m}v_0} \Rightarrow 4\frac{\kappa}{m}v_0 = \frac{1 - \frac{5}{22}}{\frac{5}{22}} = \frac{17}{5}$. Solving for κ

yields $\kappa = \frac{17}{5}\frac{m}{4v_0} = \frac{17}{5}\frac{3000}{32}\cdot\frac{1}{4} \div \left(220\left(\frac{5280}{3600}\right)\right) \approx .247\frac{\text{lb}\cdot\text{sec}^2}{\text{ft}^2}$.

For the distance traveled,

$$d = \int_0^4 v(t)dt = v_0\int_0^4 \frac{dt}{1+\frac{\kappa v_0}{m}t} = v_0\int_0^4 \frac{dt}{1+\frac{17}{20}t} = v_0\left(\frac{20}{17}\right)\ln\left(1+\frac{17}{20}t\right)\Bigg|_0^4$$

$$= 220\left(\frac{5280}{3600}\right)\left(\frac{20}{17}\right)\ln\left(1+\frac{17}{5}\right) = 562.4 \text{ ft.}$$

4. $mv' + kv = -mg, v(0) = v_0 \Rightarrow v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}$.

Set $v = 0: \frac{mg}{k} = \frac{mg}{k}\left(1+\frac{kv_0}{mg}\right)e^{-\frac{k}{m}t_m} \Rightarrow \frac{k}{m}t_m = \ln\left(1+\frac{kv_0}{mg}\right) \Rightarrow t_m = \frac{m}{k}\ln\left(1+\frac{kv_0}{mg}\right)$.

5. $h = \int_0^{t_m} v(t)dt = \int_0^{t_m}\left[-\frac{mg}{k} + \left(v_0+\frac{mg}{k}\right)e^{-\frac{k}{m}t}\right]dt = \left[-\frac{mg}{k}t - \frac{m}{k}\left(v_0+\frac{mg}{k}\right)e^{-\frac{k}{m}t}\right]\Bigg|_0^{t_m}$

$= -\frac{mg}{k}t_m + \frac{m}{k}\left(v_0+\frac{mg}{k}\right)\left(1-e^{-\frac{k}{m}t_m}\right)$.

6. $m\frac{dv}{dt} = -mg + kv^2$. $\frac{dv}{dt} \to v\frac{dv}{dy}$. Therefore, $mv\frac{dv}{dy} = mg + kv^2, v(y_0) = 0$.

Terminal velocity: $v_\infty = -\sqrt{\frac{mg}{k}}$. Solving the separable differential equation:

$m\int \frac{vdv}{kv^2 - mg} = \frac{m}{2k}\ln\left|v^2 - \frac{mg}{k}\right| = y + C, C = -y_0 + \frac{m}{2k}\ln\left(\frac{mg}{k}\right)$. Setting $v = \frac{1}{2}v_\infty = -\frac{1}{2}\sqrt{\frac{mg}{k}}$ and solving for y: $y = y_0 - \frac{m}{2k}\ln\left(\frac{4}{3}\right)$. Therefore, $\Delta y = y_0 - \left(y_0 - \frac{m}{2k}\ln\left(\frac{4}{3}\right)\right) = \frac{m}{2k}\ln\left(\frac{4}{3}\right)$.

7. $m\frac{dv}{dt} = -mg$. Thus $v(t) = -gt + C$, $v(0) = 0 \Rightarrow C = 0$. Integration then gives us $y(t) = -\frac{g}{2}t^2 + D, y(0) = y_0 \Rightarrow D = y_0$. Setting $y(t) = 0$ and solving for t yields $t^2 = \frac{2}{g}y_0$, and so the impact time $t^* = \sqrt{2y_0/g}$. Substituting this value into our equation for v gives us the impact velocity: $v(t^*) = -\sqrt{2y_0 g}$.

8. $\frac{dv}{dt} = -\frac{k}{m}x^2 v \Rightarrow v\frac{dv}{dx} = -\frac{k}{m}x^2 v \Rightarrow \frac{dv}{dx} = -\frac{k}{m}x^2 \Rightarrow v = -\frac{k}{m}\frac{x^3}{3} + C$. When $x = 0$, $v = v_0$.

Therefore, $v_0 = C$, and so $v = -\frac{k}{m}\frac{x^3}{3} + v_0$ and $x_f^{\,3} = 3\frac{m}{k}v_0 \Rightarrow x_f = \left(3\frac{m}{k}v_0\right)^{\frac{1}{3}}$.

9. $mv\dfrac{dv}{dx} = -kxv^2 \Rightarrow \dfrac{dv}{dx} = -\dfrac{k}{m}xv \Rightarrow \dfrac{dv}{dx} + \dfrac{k}{m}xv = 0$, which is a first order linear DE.

$\dfrac{d}{dx}\left(e^{\frac{k}{m}\frac{x^2}{2}}v\right) = 0 \Rightarrow v = Ce^{-\frac{kx^2}{2m}}, C = v_0 \Rightarrow v = v_0 e^{-\frac{kx^2}{2m}}$. Since $v > 0, 0 \le x < \infty, x_f = \infty$.

10. $mv\dfrac{dv}{dx} = -ke^{-x} \Rightarrow v\dfrac{dv}{dx} + \dfrac{k}{m}e^{-x} = 0 \Rightarrow \dfrac{v^2}{2} - \dfrac{k}{m}e^{-x} = C$. Then $C = \dfrac{{v_0}^2}{2} - \dfrac{k}{m}$, and so

$v^2 = 2\left[\dfrac{{v_0}^2}{2} - \dfrac{k}{m} + \dfrac{k}{m}e^{-x}\right] \Rightarrow v = \left[{v_0}^2 - 2\dfrac{k}{m}\left(1 - e^{-x}\right)\right]^{\frac{1}{2}}$. If ${v_0}^2 \ge \dfrac{2k}{m}$, then v>0 for all nonnegative x and $x_f = \infty$. If ${v_0}^2 < \dfrac{2k}{m}$, then we have ${v_0}^2 = \dfrac{2k}{m}\left(1 - e^{-x_f}\right)$, which, solved for x_f, yields $x_f = -\ln\left(1 - \dfrac{m{v_0}^2}{2k}\right)$.

11. $mv\dfrac{dv}{dx} = -\dfrac{kv}{1+x} \Rightarrow \dfrac{dv}{dx} = -\dfrac{k}{m}\left(\dfrac{1}{1+x}\right) \Rightarrow v = -\dfrac{k}{m}\ln(1+x) + C, v_0 = C$.

Therefore, $v = v_0 - \dfrac{k}{m}\ln(1+x)$ and $\dfrac{mv_0}{k} = \ln(1+x_f) \Rightarrow x_f = e^{mv_0/k} - 1$.

12. $m\dfrac{dv}{dt} + kv^2 = 0, v(0) = v_0, x(0) = 0$. We want to find v when x=d. $mv\dfrac{dv}{dx} + kv^2 = 0 \Rightarrow \dfrac{dv}{dx} + \dfrac{k}{m}v = 0 \Rightarrow v = Ce^{-\frac{k}{m}x}$. From the initial condition, $v = v_0 e^{-\frac{k}{m}x}$, and so at x=d, $v = v_0 e^{-\frac{k}{m}d}$.

13 (a). $m\dfrac{dv}{dt} = -mg + Kv^2;\ v\left(y_0\right) = 0$. We separate the variables and obtain $\dfrac{mvv'}{Kv^2 - mg} = 1$. Integrating both sides of this equation yields $\dfrac{m}{2K}\ln\left|Kv^2 - mg\right| = y + C$. From our boundary condition, we substitute and solve for C: $C = \dfrac{m}{2K}\ln mg - y_0$. Thus our explicit solution is $\dfrac{m}{2K}\ln\left|Kv^2 - mg\right| = y + \dfrac{m}{2K}\ln mg - y_0$. Since the object is falling, we know that $Kv^2 - mg < 0$, so we can rewrite this solution without the absolute value bars: $\dfrac{m}{2K}\ln\left(mg - Kv^2\right) = y + \dfrac{m}{2K}\ln mg - y_0$. With a little algebra, we can simplify this to read: $Kv^2 = mg\left[1 - e^{2\frac{K}{m}(y - y_0)}\right]$. At $y = 0$, we have $K\left(v(0)\right)^2 = mg\left[1 - e^{-2\frac{Ky_0}{m}}\right]$ and the impact velocity is $\sqrt{\dfrac{mg}{K}}\sqrt{1 - e^{-2Ky_0/m}}$.

13 (b). $v_\infty(\text{terminal velocity}) = \sqrt{mg/K} = 120$. Substitution gives us

$v(0) = 90 = \sqrt{mg/K}\sqrt{1-e^{-2Ky_0/m}} = 120\sqrt{1-e^{-2\frac{Ky_0}{m}}}$, and we can simplify this to $-2\frac{Ky_0}{m} = \ln\left(\frac{7}{16}\right)$.

Converting $\left(120\, miles/hour\right)\left(5280\, feet/mile\right)\left(\frac{1}{3600}\, hours/second\right) = 176\, feet/second$

$176 = \sqrt{\frac{mg}{K}} = \sqrt{32\frac{m}{K}}$, and solving for $\frac{K}{m}$ yields $\frac{K}{m} = \frac{32}{176^2}$. Substitution gives us $-2\frac{32}{176^2}y_0 = \ln\left(\frac{7}{16}\right)$, and solving for y_0 yields $y_0 \approx 400.11$ feet.

14 (a). $m\frac{dv}{dt} = -mg - kv \Rightarrow mv\frac{dv}{dy} = -mg - kv$. Solving the differential equation yields

$\frac{m}{k}\left(v - \frac{mg}{k}\ln\left|v + \frac{mg}{k}\right|\right) = -y + C$. Setting $v = 0$ and $y = y_0$, we

obtain: $C = y_0 - \frac{1}{g}\left(\frac{mg}{k}\right)^2\ln\left(\frac{mg}{k}\right)$. Therefore, $y_0 - y = \frac{m}{k}v - \frac{1}{g}\left(\frac{mg}{k}\right)^2\ln\left|1 + \frac{kv}{mg}\right|$.

14 (b). $v_\infty(\text{terminal velocity}) = -\frac{mg}{k}$. Set $y = 0, v = v_{impact}, v_\infty = -176\, feet/sec, v_{impact} = -132\, feet/sec$

$y_0 = \frac{1}{g}\left(\frac{mg}{k}v_{impact} - \left(\frac{mg}{k}\right)^2\ln\left|1 + \frac{kv_{impact}}{mg}\right|\right) \approx 615.93\, feet$.

15 (a). $mv\frac{dv}{dx} + \kappa_0 xv^2 = 0, v = v_0$ when $x = 0$.

15 (b). $\frac{dv}{dx} + \frac{\kappa_0}{m}xv = 0 \Rightarrow \left(e^{\frac{\kappa_0 x^2}{2m}}v\right)' = 0 \Rightarrow v = v_0 e^{-\frac{\kappa_0 x^2}{2m}}$. Setting $x = d$ and $v = 0.01v_0$, we

have $0.01v_0 = v_0 e^{-\frac{\kappa_0 d^2}{2m}} \Rightarrow \frac{\kappa_0 d^2}{2m} = \ln 100$. Solving for κ_0 yields $\kappa_0 = \frac{2m}{d^2}\ln 100$.

16 (a). $mv\frac{dv}{dr} = -\frac{GmM_e}{r^2} + \kappa v^2 \Rightarrow \frac{dv}{dr} = \frac{\kappa}{m}v - \frac{GM_e}{r^2}v^{-1}, v = 0$ when $r = R_e + h$.

16 (b). Bernoulli equation: $1 - n = -1 \Rightarrow n = 2, u = v^2 \Rightarrow v = u^{\frac{1}{2}} \Rightarrow \frac{dv}{dr} = \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dr} = \frac{\kappa}{m}u^{\frac{1}{2}} - \frac{GM_e}{r^2}u^{-\frac{1}{2}}$

$\Rightarrow \frac{du}{dr} = \frac{2\kappa}{m}u - \frac{2GM_e}{r^2}$. Therefore,

$$\left(e^{-\frac{2\kappa}{m}r}u\right)' = 2GM_e\frac{e^{-\frac{2\kappa}{m}r}}{r^2} \Rightarrow e^{-\frac{2\kappa}{m}(R_e+h)}u\Big|_{r=R_e+h} - e^{-\frac{2\kappa}{m}(R_e)}u\Big|_{r=R_e} - 2GM_e\int_{R_e}^{R_e+h}\frac{e^{-\frac{2\kappa}{m}r}}{r^2}dr$$

$$= 0 - e^{-\frac{2\kappa}{m}(R_e)}u\Big|_{r=R_e} - 2GM_e\int_{R_e}^{R_e+h}\frac{e^{-\frac{2\kappa}{m}r}}{r^2}dr.$$

Since $u = v^2, v = \dfrac{dr}{dt} < 0, v_{impact} = -e^{\frac{\kappa}{m}(R_e)}\left[2GM_e\int_{R_e}^{R_e+h}\dfrac{e^{-\frac{2\kappa}{m}r}}{r^2}dr\right]^{\frac{1}{2}}$.

Let $r = R_e + s$. Then $v_{impact} = -\left[2GM_e\int_0^h \dfrac{e^{-\frac{2\kappa}{m}s}}{(R_e+s)^2}ds\right]^{\frac{1}{2}}$.

17 (a). $v' = -g, v_0 = 0 \Rightarrow v = -gt = y' \Rightarrow y = -\dfrac{1}{2}gt^2 + y_0$. We want to find the time t at which y=7.

Thus $7 = -\dfrac{32}{2}t^2 + 555$, and solving for t yields $t \approx 5.852$ sec. At that time,

$v = -32(5.852) \approx -187.3$ ft/sec.

17 (b). $mv' + kv = -mg \Rightarrow v' + \dfrac{kv}{m} = -g, v_0 = 0$. Thus $\left(ve^{\frac{k}{m}t}\right)' = -ge^{\frac{k}{m}t} \Rightarrow ve^{\frac{k}{m}t} = -\dfrac{mg}{k}e^{\frac{k}{m}t} + C$. From

the initial condition, we have $C = \dfrac{mg}{k}$, and so

$v = -\dfrac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) \Rightarrow y = y_0 + \int_0^t v(s)ds = y_0 - \dfrac{mg}{k}\left(s + \dfrac{m}{k}e^{-\frac{k}{m}s}\right)\Bigg|_0^t$

$= y_0 - \dfrac{mg}{k}\left(t + \dfrac{m}{k}\left(e^{-\frac{k}{m}t} - 1\right)\right)$. $m = \dfrac{5\frac{1}{8}/16}{32} = \dfrac{41}{8\cdot 16\cdot 32}$ slug,

so $\dfrac{m}{k} = \dfrac{41}{8(16)(32)(0.0018)} \approx 5.56098 \text{ sec}^{-1}$.

$mg = \dfrac{41}{8(16)} \approx 0.3203125$ lb, and so solving for t yields

$7 = 555 - 177.95139\left(t - 5.56098\left[1 - e^{\frac{-t}{5.56098}}\right]\right) \Rightarrow t = 7.08513$ sec. Substitution gives us

$v = \dfrac{-0.3203125}{0.0018}\left[1 - e^{\frac{-7.08513}{5.56098}}\right] \approx -128.18$ ft/sec.

18. $mg = 180\, lb$. For $0 \le t \le 10, v' = -g,\ v(0) = 0$.

For $10 < t \le 14, mv' + kv = -mg,\ y(14) = 0$.

For $mg = 200,\ \dfrac{200}{\text{k}} = 10\dfrac{5280}{3600} \Rightarrow k = \dfrac{3600(200)}{5280(10)} = 13.63636364$.

18 (a). $v=-gt$ At $t=10, v=-320\ ft/\sec$.

18 (b). Solve $v'+\dfrac{k}{m}v=-g, v(0)=-320$, for $v(4)$.

$$v(t)=-\frac{mg}{k}+\left(v_0+\frac{mg}{k}\right)e^{-\frac{k}{m}t}\Rightarrow v(4)=-\frac{180}{13.\overline{63}}+\left(-320+\frac{180}{13.\overline{63}}\right)e^{-\frac{13.\overline{63}(32)}{180}(4)}$$

$=-13.2-306.8(.000061469)=-13.219\ ft/\sec$ (basically the terminal velocity).

18 (c). $$h=-\int_0^4 v(t)dt=\left(\frac{mg}{k}t-\left[v_0+\frac{mg}{k}\right]\left(-\frac{m}{k}\right)e^{-\frac{k}{m}t}\right)\Bigg|_0^4=\frac{mg}{k}(4)+\frac{m}{k}\left(v_0+\frac{mg}{k}\right)\left(e^{-\frac{4k}{m}}-1\right)$$

$$=\frac{4mg}{k}-\frac{m}{k}\left(v_0+\frac{mg}{k}\right)\left(1-e^{-\frac{4k}{m}}\right)=\frac{4(180)}{13.\overline{63}}-\frac{180}{32(13.\overline{63})}\left(-320+\frac{180}{13.\overline{63}}\right)\left(1-e^{-\frac{4(13.\overline{63})32}{180}}\right)$$

$=52.8-0.4125(-306.8)(0.99994)=179.347\ ft$.

18 (d). $h_{balloon}=h+\dfrac{1}{2}g(10)^2=179.347+1600=1779.347\ ft$.

19. For the first situation, $mv_1'+kv_1=0, v_1=v_0e^{-\frac{k}{m}t}, m=\dfrac{3000}{32}, k=25$. Then

$$\frac{50}{220}=e^{\frac{-25\cdot 32}{3000}t_1}\Rightarrow t_1=\frac{3000}{25(32)}\ln\frac{22}{5}\approx 5.556\ \text{sec.}$$

For the second situation, $mv_2'+k(\tanh t)v_2=0, v_2'+\dfrac{k}{m}\tanh t\left(v_2\right)=0$. This is a first order linear

equation. Letting $\mu=e^{\frac{k}{m}\ln(\cosh t)}=\left(\cosh t\right)^{\frac{k}{m}}$, we have $\left(v_2\left(\cosh t\right)^{\frac{k}{m}}\right)'=0\Rightarrow v_2=C\left(\cosh t\right)^{-\frac{k}{m}}$.

From the initial condition, we have $\cosh(0)=1\Rightarrow C=v_0$.

Then $\dfrac{v_2}{v_0}=\left(\cosh t\right)^{-\frac{k}{m}}\Rightarrow \cosh t_2=\left(\dfrac{v_0}{v_2}\right)^{\frac{m}{k}}=\left(\dfrac{220}{50}\right)^{\frac{3000}{32\cdot 25}}$.

$\ln(\cosh t_2)=3.75\ln\left(\dfrac{22}{5}\right)\approx 5.55602\Rightarrow \cosh t_2\approx 258.79$, so $t_2\approx\cosh^{-1}(258.79)\approx 6.249$ sec.

This would be expected, since the size of the drag coefficient would be less for the second situation. Comparing the two values gives us $t_1\approx 0.89t_2$. These values do not seem appreciably different. However, it can be shown that this difference in stopping time leads to a difference in stopping distance of approximately 110 ft. If this distance is important for a certain situation, then the idealization is not reasonable.

20. $mv' = -mg + \kappa v^2, v(0) = 0 \Rightarrow v' = -g + \frac{\kappa}{m}v^2 = \frac{\kappa}{m}\left(v^2 - \frac{mg}{\kappa}\right) \frac{v'}{v^2 - \frac{mg}{\kappa}} = \frac{\kappa}{m}$

$$\frac{1}{v^2 - \frac{mg}{\kappa}} = \frac{A}{v - \sqrt{\frac{mg}{\kappa}}} + \frac{B}{v + \sqrt{\frac{mg}{\kappa}}} \Rightarrow A = \frac{1}{2\sqrt{\frac{mg}{\kappa}}}, B = -\frac{1}{2\sqrt{\frac{mg}{\kappa}}}.$$

Therefore, $\frac{1}{2\sqrt{\frac{mg}{\kappa}}}\ln\left|\frac{v - \sqrt{\frac{mg}{\kappa}}}{v + \sqrt{\frac{mg}{\kappa}}}\right| = \frac{\kappa}{m}t + C, v(0) = 0 \Rightarrow C = 0$ and $-\sqrt{\frac{mg}{\kappa}} < v \le 0$.

Then, $\left|\frac{v - \sqrt{\frac{mg}{\kappa}}}{v + \sqrt{\frac{mg}{\kappa}}}\right| = \frac{\sqrt{\frac{mg}{\kappa}} - v}{\sqrt{\frac{mg}{\kappa}} + v} = e^{2\sqrt{\frac{\kappa g}{m}}t} \Rightarrow v = -\sqrt{\frac{mg}{\kappa}}\left(\frac{1 - e^{-2\sqrt{\frac{\kappa g}{m}}t}}{1 + e^{-2\sqrt{\frac{\kappa g}{m}}t}}\right) = -\sqrt{\frac{mg}{\kappa}}\tanh\left(\sqrt{\frac{\kappa g}{m}}t\right)$.

21. $10\ mi/hr = 10\left(\frac{5280}{3600}\right) = 14.67$ ft/sec. Then $14.67 = \sqrt{\frac{200}{\kappa}} \Rightarrow \kappa \approx .929 \frac{\text{lb}\cdot\text{sec}^2}{\text{ft}^2}$.

22 (a). $m\ell^2\theta'' = -mg\ell\sin\theta = m\ell^2\frac{d\omega}{dt} \Rightarrow m\ell^2\omega\frac{d\omega}{d\theta} = -mg\ell\sin\theta$

$m\ell^2\omega\frac{d\omega}{d\theta} = -mg\ell\sin\theta$ and $\omega = -\omega_0$ when $\theta = \theta_0$.

22 (b). $m\ell^2\frac{\omega^2}{2} = mg\ell\cos\theta + C,\ m\ell^2\frac{\omega_0^{\ 2}}{2} = mg\ell\cos\theta_0 + C$

$$\Rightarrow m\ell^2\frac{\omega^2}{2} - mg\ell\cos\theta = m\ell^2\frac{\omega_0^{\ 2}}{2} - mg\ell\cos\theta_0$$

22 (c). When $\theta = 0,\ m\ell^2\frac{\omega^2}{2} - mg\ell = m\ell^2\frac{\omega_0^{\ 2}}{2} - mg\ell\cos\theta_0$

$$\Rightarrow \omega^2 = \left(\frac{2}{m\ell^2}\right)\left(m\ell^2\frac{\omega_0^{\ 2}}{2} + mg\ell - mg\ell\cos\theta_0\right)$$

$$\Rightarrow \omega = \sqrt{\omega_0^2 + \frac{2g}{\ell}(1 - \cos\theta_0)}.$$

23. $m\ell^2\frac{\omega^2}{2} = mg\ell\cos\theta + C,\ \omega = \omega_0$ when $\theta = 0$. Therefore, $C = m\ell^2\frac{\omega_0^{\ 2}}{2} - mg\ell$, and so

$m\ell^2\frac{\omega^2}{2} = mg\ell\cos\theta + m\ell^2\frac{\omega_0^{\ 2}}{2} - mg\ell$. We know that $\omega = 0$ when $\theta = \frac{3\pi}{4}$, so $-\dfrac{mg\ell}{\sqrt{2}}+\dfrac{m\ell^2\omega_0{}^2}{2}-mg\ell=0\Rightarrow\omega_0{}^2=\dfrac{2}{m\ell^2}mg\ell\left(1+\dfrac{1}{\sqrt{2}}\right)=\dfrac{g}{\ell}\left(2+\sqrt{2}\right).$

Thus $\omega_0=\sqrt{\dfrac{g}{\ell}\left(2+\sqrt{2}\right)}=\sqrt{16\left(2+\sqrt{2}\right)}\approx 7.391$ rad/sec.

Section 2.10

Note: for exercises 1-5, h=0.1

1 (a). $y_{k+1}=y_k+h\left(2t_k-1\right),t_0=1,y_0=0$

1 (b). $y_1=0.1,y_2=0.22,y_3=0.36$

1 (c). $y=t^2-t+C,y(1)=C=0\Rightarrow y=t^2-t$

2 (a). $y_{k+1}=y_k-hy_k,t_0=0,y_0=1$

2 (b). $y_1=0.9,y_2=0.81,y_3=0.729$

2 (c). $y=Ce^{-t},y(0)=C=1\Rightarrow y=e^{-t}$

3 (a). $y_{k+1}=y_k-h\left(t_ky_k\right),t_0=0,y_0=1$

3 (b). $y_1=1,y_2=0.99,y_3=0.9702$

3 (c). $y=Ce^{-\frac{t^2}{2}},y(0)=C=1\Rightarrow y=e^{-\frac{t^2}{2}}$

4 (a). $y_{k+1}=y_k+h\left(-y_k+t_k\right),t_0=0,y_0=0$

4 (b). $y_1=0,y_2=0.01,y_3=0.029$

4 (c). $y=Ce^{-t}+t-1,y(0)=C-1=0\Rightarrow y=e^{-t}+t-1$

5 (a). $y_{k+1}=y_k+h\left(y_k{}^2\right),t_0=0,y_0=1$

5 (b). $y_1=1.1,y_2=1.221,y_3=1.3700841$

5 (c). $y^{-2}y'=1,-y^{-1}=t+C,C=-1\Rightarrow y=\dfrac{1}{1-t}$

6 (a). $y_{k+1}=y_k+hy_k,t_0=-1,y_0=-1$

6 (b). $y_1=-1.1,\ y_2=-1.21,\ y_3=-1.331$

6 (c). $y=Ce^t,\ C=-e\Rightarrow y=-e^{1+t}$

11 (a). (i)Euler's method will underestimate the exact solution.

(ii)Euler's method will overestimate the exact solution.

(iii)Euler's method will underestimate the exact solution.

(iv)Euler's method will overestimate the exact solution.

11 (b). Euler's method should initially underestimate (when solution curves are concave up) and then tend to "catch up" (when solution curves become concave down).

12. $y_{k+1} = y_k + h\left(t_k y_k + \sin\left(2\pi t_k\right)\right), y_0 = 1, h = 0.01, k = 0,1,\ldots,99$.

13. $V(0) = 90, V(t) = 90 + 5t, V(T) = 100$ when $T = 2 \Rightarrow 0 \le t \le 2$

$$\frac{dQ}{dt} = 6\left(2 - \cos(\pi t)\right) - 1 \cdot \frac{Q}{90 + 5t}, Q(0) = 0$$

$$Q_{k+1} = Q_k + h\left[6\left(2 - \cos(\pi t_k)\right) - \frac{Q_k}{90 + 5t_k}\right], Q_0 = 0, h = 0.01, k = 0,1,2,,\ldots,199$$

Result: $Q(2) = 23.7556..oz.$

14. $P' = 0.1\left(1 - \frac{P}{3}\right)P + e^{-t}, P(0) = \frac{1}{2}$. $P_{k+1} = P_k + h\left[0.1\left(1 - \frac{1}{3}P_k\right)P_k + e^{-t_k}\right], P_0 = 0.5$.

With $h = 0.01, k = 0,1,\ldots,199, t_k = 0.01k$, $P(2) = 1.502477$ million.

15 (a). $y_{k+1} = y_k + h(y_k + 1), y_0 = 0$. For $y_k^{(1)}, h = 0.02, k = 0,1,\ldots 49$

For $y_k^{(2)}, h = 0.01, k = 0,1,\ldots 99$.

15 (b). $y = Ce^t - 1, C = 1 \Rightarrow y = e^t - 1$.

16 (a). $y' - \lambda y = 0, \left(e^{-\lambda t} y\right)' = C, y = Ce^{\lambda t}, y(0) = C = y_0$. Thus $y = e^{\lambda t} y_0$.

16 (b). $y_{k+1} = y_k + h\lambda y_k = \left(1 + \lambda h\right) y_k$.

Therefore $y_1 = \left(1 + \lambda h\right) y_0, y_2 = \left(1 + \lambda h\right) y_1 = \left(1 + \lambda h\right)^2 y_0, y_n = \left(1 + \lambda h\right)^n y_0$,

16 (c). $y_n = \left(1 + \frac{\lambda t}{n}\right)^n y_0$. Since $\lim_{n \to \infty}\left(1 + \frac{a}{n}\right)^n = e^a$, the result follows.

17 (a). $y' = 2t - 1,\ y(1) = 0, y(t) = t^2 - t + C,\ y(1) = C = 0$. Thus $y = t^2 - t$.

17 (b). $y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+h}} (2s - 1)ds = y(t_k) + \left(t_k + h\right)^2 - t_k^2 - h = y(t_k) + 2t_k h + h^2 - h$

For Euler's Method: $y_{k+1} = y_k + h(2t_k - 1) = y_k + 2t_k h - h$. Therefore, Euler's Method will not produce exact values.

17 (c). For R-K, $y_{k+1} = y_k + \frac{h}{6}(12t_k + 6h - 6) = y_k + 2t_k h + h^2 - h$ .Therefore, R-K algorithm will generate exact values.

18 (a). $y_1^E = 0.9, y_1^{RK} = 0.9048375$

18 (b). $y(t) = e^{-t}$

19 (a). $y_1^E = 1.0000, y_1^{RK} = 0.9950$

19 (b). $y(t) = e^{-t^2/2}$

20 (a). $y_1^E = 0, y_1^{RK} = 0.0048375$

20 (b). $y(t) = t - 1 + e^{-t}$

21 (a). $y_1^E = 1.1000, y_1^{RK} = 1.1111...$

21 (b). $y(t) = \dfrac{1}{1-t}$

22 (a). $y_1^E = -1.1, y_1^{RK} = -1.10517083$

22 (b). $y(t) = -e^{1+t}$

Review Exercises

2. $y(t) = Ce^{t^3+12t}$

4. $y(t) = \left(e^{3t} - 1\right)^{\frac{1}{3}}$

6. $y(t) = 3e^{2\sqrt{t}}, t > 0$

8. $y(t) = 5 - \cos t$

10. $y(t) = Ce^{2t}$

12. $y(t) = -3e^{2t} + Ce^{4t}$

14. $y(t) = -\dfrac{1}{2}\ln\left(\dfrac{5}{3} - \dfrac{2}{3}e^{3t}\right)$

16. $y(t) = \begin{cases} 3te^t, 0 \le t < 1 \\ 3e^t,\ 1 \le t \le 2 \end{cases}$

18. $y(t) = Ct^2$

20. $y(t) = \left(Ce^{3t^2} - 1\right)^{\frac{1}{3}}, C > 0$

22. $y(t) = \dfrac{-3t^3 \pm \sqrt{9t^6 - 4C}}{2}$

24. $y(t) = \pm\sqrt{\ln(e^t + C)}$

26. $y(t) = \pm\sqrt{C - \frac{t^2}{5}}$

28. $t^2\left(y + 3y^3\right) = C$

30. $y(t) = \cos^{-1}\left(C - \frac{1}{t}\right), t > 0$