Instructors’ Manual
to accompany
FOUNDATIONS OF GEOMETRY

Updated June 2006

Gerard A. Venema
Department of Mathematics and Statistics
Calvin College

Upper Saddle River, New Jersey 07458
## Contents

Website and Lab Manual .................................................. v

Teaching from THE FOUNDATIONS OF GEOMETRY ............... 1

Comments on individual chapters ................................. 4

Solutions to Exercises in Chapter 1 ................................. 11

Solutions to Exercises in Chapter 2 ................................. 16

Solutions to Exercises in Chapter 3 ................................. 19

Solutions to Exercises in Chapter 4 ................................. 24

Solutions to Exercises in Chapter 5 ................................. 27

Solutions to Exercises in Chapter 6 ................................. 44

Solutions to Exercises in Chapter 7 ................................. 73

Solutions to Exercises in Chapter 8 ................................. 95

Solutions to Exercises in Chapter 9 ................................. 113

Solutions to Exercises in Chapter 10 ................................ 129

Solutions to Exercises in Chapter 12 ................................ 141

Solutions to Exercises in Chapter 13 ................................ 169

Solutions to Exercises in Chapter 14 ................................ 181

Bibliography ................................................................. 187
Website and Lab Manual

WEBSITE
The author maintains a website for the textbook. The URL is

http://calvin.edu/~venema/geometrybook.html.

The website contains current information about the book, supplementary materials, and errata.

GEOMETER’S SKETCHPAD LAB MANUAL
A computer lab manual, entitled Exploring Advanced Euclidean Geometry with Geometer’s Sketchpad, is available at the website described above. The manual is designed to supplement the textbook. It contains complete instructions for students who are just getting started with Geometer’s Sketchpad and goes on to more advanced techniques such as custom tools and action buttons. The mathematical content greatly expands on the explorations in sections 7.7 and 10.7 of the textbook. There are also chapters on Euclidean inversions and the Poincaré disk model. The manual may be downloaded as a collection of pdf files.

CONTACT ME
Please contact me if you have any comments or questions regarding either the textbook, the Instructors’ Manual, or the GSP supplement.

Gerard A. Venema
venema@calvin.edu
June, 2006
Teaching from THE FOUNDATIONS OF GEOMETRY

The following suggestions for teaching from THE FOUNDATIONS OF GEOMETRY are quite subjective. They are presented in the hope that they will be useful to some instructors, but they are not meant to be prescriptive. This first chapter contains a few general comments and a table showing topic dependencies. The second chapter contains comments about how to cover individual chapters from the book.

GETTING STARTED

The main consideration in getting the course off the ground is that you need to get to Chapters 5 and 6 as quickly as possible. Chapters 1 through 4 should be treated as introductory and the course should not be allowed to get bogged down in them. The comments below on individual chapters contain specific suggestions about how to achieve this.

One thing the students may not understand at the outset is what it means to study the foundations of geometry and why understanding the foundations might be an important goal. The Van Hiele model is useful in helping to explain this. Ideally students entering the course should be at Level 3 when they enter the course and the course should bring them to Level 4. (Assuming the steps are numbered as in Appendix D.) Of course some students are not at Level 3 when they enter the course. The first five chapters are designed to help bring students to that level.

SUPPLEMENTARY READING

It is suggested that students read something about the history of geometry while working through this book. Euclid’s Window [5] has a terrible reputation (see [4]), but it is really not such a bad choice—for this particular course. It is the only book I have found that is exclusively devoted to geometry and which describes the various revolutions that have taken place in the conventional understanding of the relationship between the theorems of geometry and the real world. The chapter on string theory is rather thin. The book is written in a style that students will appreciate and enjoy even if you don’t.

Choose one of the books listed in the suggested readings at the end of the chapters and stick with it. Assign readings from your selected book as the topics arise.

CONTINUITY

There are occasional proofs that involve \((\epsilon, \delta)-arguments\). If your class is not comfortable with such proofs, they can be omitted without serious consequence. No
great harm is done by simply accepting the continuity results as additional axioms.

**DEPENDENCIES OF TOPICS**

<table>
<thead>
<tr>
<th>Section</th>
<th>Topic</th>
<th>Prerequisites</th>
<th>Needed for</th>
</tr>
</thead>
<tbody>
<tr>
<td>§5.7</td>
<td>Continuity Axiom</td>
<td>Crossbar Theorem</td>
<td>Circular Continuity (§10.5)</td>
</tr>
<tr>
<td>§6.4</td>
<td>Continuity of Distance</td>
<td>Triangle Inequality</td>
<td>Dissection Theorem (9.4.4), Bolyai’s Theorem (9.5.1), and Elementary Circular Continuity (10.2.8)</td>
</tr>
<tr>
<td>§6.9</td>
<td>Properties of Saccheri and Lambert quadrilaterals</td>
<td>§§6.7, 6.9</td>
<td>Chapter 8</td>
</tr>
<tr>
<td>§7.7</td>
<td>Exploring geometry of the triangle</td>
<td>§§7.1–7.5</td>
<td>§10.7</td>
</tr>
<tr>
<td>§8.2</td>
<td>Theorem 8.2.12</td>
<td>§8.2</td>
<td>Bolyai’s Thm. (9.5.1)</td>
</tr>
<tr>
<td>§8.5</td>
<td>Properties of angle of parallelism</td>
<td>§8.4</td>
<td>Properties of defect (§8.6) and Bolyai’s Theorem (9.5.1)</td>
</tr>
<tr>
<td>§9.3</td>
<td>Neutral dissection theory</td>
<td>§9.1</td>
<td>§§9.4, 9.5</td>
</tr>
<tr>
<td>§9.4</td>
<td>Euclidean dissection theory</td>
<td>§§9.2, 9.3</td>
<td>§9.5</td>
</tr>
<tr>
<td>§9.5</td>
<td>Hyperbolic dissection theory and Bolyai’s Theorem</td>
<td>§9.4, §8.2, and Continuity of defect (8.6.5)</td>
<td>Not used elsewhere</td>
</tr>
<tr>
<td>§§10.1–10.3</td>
<td>Circles in neutral geometry</td>
<td>Chapter 6</td>
<td>§10.4, Chapter 11</td>
</tr>
<tr>
<td>§10.4</td>
<td>Circles in Euclidean geometry</td>
<td>§§10.1–10.3, 7.1–7.5</td>
<td>§§10.7, 11.3, 12.7</td>
</tr>
<tr>
<td>§10.5</td>
<td>Circular continuity</td>
<td>§§10.1–10.3</td>
<td>Chapter 11</td>
</tr>
<tr>
<td>§10.6</td>
<td>Euclidean area and circumference</td>
<td>§§7.4, 9.2, 10.2</td>
<td>Not used elsewhere</td>
</tr>
<tr>
<td>§10.7</td>
<td>Exploring Euclidean circles</td>
<td>§§7.7, 10.4</td>
<td>Not used elsewhere</td>
</tr>
<tr>
<td>Chap. 11</td>
<td>Compass &amp; straightedge constructions</td>
<td>Circular continuity (§10.5), which may be assumed as an axiom</td>
<td>Not used elsewhere</td>
</tr>
<tr>
<td>Section</td>
<td>Topic</td>
<td>Prerequisites</td>
<td>Needed for</td>
</tr>
<tr>
<td>---------</td>
<td>--------------------------------------</td>
<td>----------------------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>§12.2</td>
<td>Properties of Isometries</td>
<td>Chapter 6</td>
<td>§§12.3–12.7</td>
</tr>
<tr>
<td>§§12.3–12.5</td>
<td>Classification of rigid motions</td>
<td>§12.2</td>
<td>Not used elsewhere</td>
</tr>
<tr>
<td>§12.6</td>
<td>Reflection Postulate</td>
<td>§12.2</td>
<td>Proof that Poincaré models satisfy neutral postulates (Chap. 13)</td>
</tr>
<tr>
<td>§12.7</td>
<td>Inversions in circles</td>
<td>§§7.4, 10.4</td>
<td>Proof that Poincaré models satisfy neutral postulates (Chap. 13)</td>
</tr>
<tr>
<td>Chap. 13</td>
<td>Models</td>
<td>In order to give complete proofs that the models satisfy the neutral postulates it is necessary to cover all of §§12.6 and 12.7</td>
<td>Not used elsewhere</td>
</tr>
<tr>
<td>Chap. 14</td>
<td>Polygonal models and geometry of space</td>
<td>At least some of Chapter 8</td>
<td></td>
</tr>
</tbody>
</table>
Comments on individual chapters

CHAPTER 1
This chapter should not be omitted because it provides the background for the rest of the course. But there is no need to spend more than one or two days on it. It is important to discuss the statements of Euclid's postulates and their place in the logical arrangement of the Elements. It is also a good idea to go step-by-step through one or two of the proofs from Euclid that are included and to discuss both the proof as Euclid would have understood it and also the weaknesses that we now see in the proofs.

Most students will not be prepared to appreciate the subtleties of the critique of Euclid's proofs. It might be a good idea to review Euclid's proofs again later in connection with the neutral geometry of Chapters 5 and 6. By that time the students may have gained the necessary maturity to understand why some of Euclid's proofs are considered to be incomplete.

It will be apparent to the reader that I have a rather high regard for Euclid and his work. In particular, I think it is a mistake to tell students that Euclid's proofs are "flawed" because they do not conform to modern standards of rigor. This leads students to conclude that Euclid is not worth studying when, in fact, nothing could be further from the truth. I think it is important to try, as best we can, to understand how Euclid understood the role of axioms and to recognize (in the next chapter) that the modern understanding is different.

Much of Chapter 1 can be assigned as reading.
Suggestion: After the students have tried to work Exercise 1.10 by hand, have them draw the diagram using Geometer's Sketchpad. They can vary the shape of the triangle and see that the diagrams shown in the text never actually occur.

CHAPTER 2
This chapter is also important to what comes later and should not be omitted. But again you should not spend too much time on the chapter. One or two days should suffice. Incidence geometry is a kind of "laboratory" in which to try out some of the ideas in the course. It is important for the course in two ways. First, it provides a setting in which to understand what it means to say that a parallel postulate is independent of the other postulates. Second, it is a setting in which to learn to write proofs and to base those proofs on what is actually stated in the postulates (and nothing else).

The philosophy of the book is that geometry should be seen as a subject that is closely tied to actual spatial relationships in the real world. In keeping with that philosophy, the only example of an axiomatic system included in the chapter
is incidence geometry. An instructor may want to include one or more examples that involve undefined terms that do not carry as much intuitive freight as the terms “point” and “line.” Something like the “Scorpling Flugs” example on pages 164–168 of [9] would be good.

CHAPTER 3

How much time you spend on this chapter will depend on the background of the students. If the students have never written proofs before, you could spend two or three days discussing some of the principles explained in the chapter. If the students already have experience writing proofs, you could simply ask them to read the chapter as review and go directly to the proofs of the theorems in incidence geometry. In any case, the theorems from incidence geometry should be discussed and several of the proofs should be assigned as exercises. In my experience most students make rather fundamental logical errors in their first attempts to prove these theorems. In particular, most students want to read Incidence Axiom 3 to say what Theorems 3.6.7 and 3.6.8 say. (See the comments after the solutions to Exercises 3.8 and 3.9.) I have not found a way to prevent students from making this error, but they seem to learn something when they make the error and it is explained to them. As a result, the experience of writing proofs of the theorems in neutral geometry can be an important step of the development of student thinking.

It might be a good idea to distinguish between the style of a written proof and the style of a proof that is communicated orally. In this chapter you will want to use class time to go line-by-line through at least one written proof, but later in the course you should not use valuable class time to present the details of written proofs. If you make this distinction clear, you will be able to require students to submit careful proofs that include complete written justifications for each step even though the proofs you present in class follow a different style.

CHAPTER 4

This chapter should be mostly review for most students and could well be assigned as reading. Try to spend no more than one or two class periods on it.

Even though you will not want to spend much class time on the chapter, there is an important reason for including it. The postulates stated in Chapter 5 will be expressed in set-theoretic terminology and will be based on properties of the real numbers. As a result it is important to review the terminology of set theory and it is essential to be clear about what is being assumed regarding the real numbers. The algebraic properties of the real numbers are familiar to students and there is no need to review them. But some of the topological properties of the real numbers are less well known to students and are certainly not well understood. In particular, students need to be reminded of the density of both the rationals and the irrationals and most calculus students have never seen the statement of the Least Upper Bound Postulate.

If nothing else, the chapter serves as a reference for basic facts about the real numbers when those facts are needed later. In particular, the proof of
the Parallel Projection Theorem in Chapter 7 requires the Comparison Theorem and the construction of the angle of parallelism in Chapter 8 requires the Least Upper Bound Postulate. The Archimedean Property of the Real Numbers is used repeatedly in constructions—the first time is in the proof of the Saccheri-Legendre Theorem in Chapter 6.

CHAPTER 5

In this chapter the foundations of the remainder of the course are laid, so the chapter should be covered carefully. At the same time, this is not where the interesting and beautiful theorems of geometry are to be found, so you don’t want to get bogged down in the chapter. Students will probably not learn to appreciate proofs in this chapter either (that will happen in the next chapter), so only a limited number of proofs should be assigned.

Section 5.1 can be assigned as reading. It is important, however, that future teachers have some idea of the kind of thinking that goes into the selection of a system of axioms as well as some knowledge of the range of options available when a system of axioms is being chosen. There is another reason for preferring metric axioms that is not emphasized in the text: essentially all modern work in geometry is done in the context of a metric. From that point of view, the axioms of Mac Lane are to be preferred over any of the other systems mentioned in the text. The decision to use the more SMSG-like axioms in the text is based on the need to make direct connections with what is done in high school geometry.

All the statements of the six neutral axioms in sections 3, 4, 5, 6, and 8 and the related definitions should be covered carefully. The examples and explanatory material in the second half of Section 5.4 are optional. They add to student understanding of what is being asserted in the Ruler Postulate, but they are not essential to the logical development of the course.

Section 5.7 contains results that are important for the later development of the course. At a minimum, students need to know the statements of the named theorems in the section and all the definitions. The proofs in the section are not difficult; the problem is that they are also not interesting. Most students find the proofs in Chapter 6 to be much more satisfying.

Section 5.7 can be treated in several different ways. It would be perfectly respectable and logically correct to simply assume the main results of the section as axioms and move on. Another possibility is to skip many of the proofs in this section temporarily and come back to them later, after the next few chapters have been studied. On the other hand, a strong argument in favor of studying this section carefully is the fact that these results are an essential part of the foundations of geometry and only by mastering them will you really come to understand the foundations. In particular, that is the only way to appreciate what is normally omitted from high school treatments of Euclidean geometry.

Most instructors will want to include some of the proofs from section 5.7 and omit others. It is, for example, quite acceptable to treat the Betweenness Theorem for Rays and the Linear Pair Theorem as two additional parts of the Protractor
Postulate. After all, exactly that is quite often done in courses at this level—see [11], [3], [8], [6], and [10], for example. The instructor can simply point to the fact that there is a proof in the book as evidence that these axioms can be proved as consequences of the others. It is not necessary for every student of the subject to verify this.

Be careful with the exercises in Chapter 5. Some of them, such as 5.5(c) and 5.13–5.19, may require mathematical maturity that the students do not yet have. Think carefully before assigning these exercises. On the other hand, some or all of exercises 5.24–5.35 should definitely be assigned.

CHAPTER 6

This chapter is the heart of the course. In the preceding chapters it was important to keep moving. Now it is time to slow down and enjoy the material in this chapter. It’s classic and beautiful mathematics, but understanding it is well within the grasp of average students.

It is in this chapter that the students learn to write good proofs and begin to really appreciate proofs. Student ability to write proofs and to construct mathematical arguments should develop significantly in this chapter. The proofs in the first part of the chapter are straightforward arguments that are good practice for what is to come. The proofs of the equivalences with the Euclidean Parallel Postulate (EPP) require a substantial increase in the level of thinking involved. Once students reach this level they feel good about the fact that they can prove so many statements are equivalent to EPP.

Try to assign as many of the omitted proofs as possible. Students come to a deep understanding of the material by working out these proofs for themselves. If you are not able to assign all of them as homework, you could discuss some of them in class. These proofs are not difficult and students learn a lot from working through the details for themselves. It may appear that some of the hints give away too much, but most of the students do not see it that way. If they know the basic outline of the proof they can concentrate on the form of the proof.

I think it is important in these early proofs that students include a justification for every step in the argument. Learning this discipline early heads off lots of sloppiness that leads to trouble later. Each instructor will have to set a standard that is appropriate for his or her own class. The solutions provided in this manual include a justification for nearly every step.

CHAPTER 7

Some coverage of this chapter should probably be included in any course for prospective high school teachers. It cannot be assumed that such students already know how to prove the Similar Triangles Theorem or that they know how to use similar triangles to prove the Pythagorean Theorem. The material in sections 7.3 through 7.5 is relatively easy and each of these sections can be covered in a day. This means that you can cover the rudiments of Euclidean Geometry in one week if you want to save time for other topics.
8 Comments on individual chapters

Sections 7.6 and 7.7 are optional. Section 7.6, on trigonometry, is included in order to make it clear that the Similar Triangles Theorem and the Pythagorean Theorem are the foundation on which trigonometry is built.

Ideally students should work through section 7.7 in a computer laboratory setting with the instructor present. An alternative would be to ask students to work through parts of it on their own. The instructor could work through it in class if the classroom is equipped with a projector for computer output, but this is less desirable than having the students do it for themselves. The proofs of the concurrence theorems and the Euler Line Theorem make great exercises. The exploration in §10.7 builds on this one, but the theorems in this section are not used anywhere else in the book.

CHAPTER 8

It should be possible to cover sections 8.2, 8.3, and most of 8.4 in about one week. Theorem 8.2.12 can be omitted unless you plan to cover the material (at the end of Chapter 9) on area and defect in hyperbolic geometry. Sections 8.5 and 8.6 require some of the techniques of analysis, but contain some very interesting theorems that make the extra effort worthwhile. Section 8.7 can be omitted or assigned as reading.

Sections 8.3 through 8.6 build understanding of hyperbolic geometry. Specifically, they aim to understand parallelism and defect. If you do not have time to cover all of this material, you should probably just start at the beginning and cover as much as you can; there are no theorems (after 8.2.12) that are natural candidates for omission. The only theorem from the second half of the chapter that is actually needed for subsequent work is Continuity of Defect (Theorem 8.6.5), but its statement is very natural and easily understood. It would, therefore, be possible to use the statement of that theorem later even if not all of the proof has been covered.

The proofs of the stability of limiting parallelism (Theorem 8.4.9) and the symmetry of limiting parallelism (Theorem 8.4.12) have been left as exercises. The hints in the back of the book give away the main idea of each of these proofs, but there are still many details (having to do with betweenness) that must be checked. Most students do not appreciate these subtleties, and will probably just assert that the relationships are obvious from the diagram. It may, therefore, be necessary to discuss some aspects of the proofs in class before they are assigned. Complete proofs are included in this instructors’ manual.

CHAPTER 9

At least some area-based proofs of the Pythagorean Theorem (section 9.3) should be included in a course for future teachers.

A quick treatment of dissection theory could include just section 9.3 followed by the simple Euclidean proof of the Dissection Theorem that is outlined in exercises 9.15–9.17.

The proof of Bolyai’s Theorem (section 9.5) explains the relationship between area and defect in hyperbolic geometry. It is a great topic to include if you have
time and working through it helps students gain a much deeper understanding of (axiomatic) hyperbolic geometry. The proof builds on the dissection theory developed in sections 9.2 and 9.4. The prerequisites from Chapter 8 are minimal: just section 8.2 and the Continuity of Defect (Theorem 8.6.5). The statement of Continuity of Defect is natural and intuitive enough that it could be used even if you have not done the complete proof.

CHAPTER 10
It should be possible to cover sections 10.1, 10.2, 10.3, and part of 10.4 in one week. The material in section 10.4 is needed for the treatment of Euclidean inversions in section 12.7.

Circular continuity should logically precede the material on constructions that comes in Chapter 11. But it is perfectly OK to accept circular continuity as an axiom when you do Chapter 11.

The material in section 10.6 should be well known to future teachers, but (unfortunately) most do not seem to be familiar with the definition of $\pi$. It seems that, for most students, $\pi$ is simply a number that appears when you push a particular button on a calculator. Thus it would be good to cover this material in a course for high school teachers—if time permits.

The exploration in section 10.7 is optional, but again it is a great place to show an appropriate use of technology in the geometry course.

CHAPTER 11
Unfortunately most one-semester courses will not have time to cover this chapter. The material is interesting and classical, so try to make room for some of it if you can. Even if the chapter is not covered in class, parts of it could be used as a source for student projects.

CHAPTER 12
Every course for future high school teachers should include some treatment of transformations. Sections 12.2 and 12.6 can be covered in one week. The classification of rigid motions of the plane (sections 12.3–12.5) is another great topic for future teachers.

The material on Euclidean inversions (section 12.7) relies on 10.4 and is needed in order to give a complete construction of the Poincaré models in Chapter 13.

The proof that the Horolation Theorem (Exercise 12.39) is probably too complicated for most students to discover on their own (although it makes a great challenge exercise for strong students). There is a complete proof in the theorem in the instructors’ manual. If you wish to give a complete proof of the classification of hyperbolic motions, you will probably want to discuss the Horolation Theorem in class rather than simply assigning it as an exercise. You could also omit that exercise and just present the material in the text. While the proof of the classification theorem in the text is not complete, it does show that the hyperbolic proof is essentially the
Comments on individual chapters

same as the Euclidean proof.

CHAPTER 13

The interpretations themselves can be described with very little background. Therefore it works to cover much of this chapter even if you have not done all of Chapters 10 and 12. It is recommended that at least some coverage of the Poincaré models be included in any course for future high school teachers.

If you want to give complete proofs that the Poincaré disk model satisfies the Ruler and Reflection Postulates, you will have to prepare the way by covering sections 10.4, 12.6 and 12.7. This means that the length of Chapter 13 is somewhat misleading; it can be relatively short only because most of the hard work is done in earlier chapters.

The proof that the transformation described on pages 337 and 338 of the text maps Klein lines to Poincaré lines (Exercise 13.12) is probably too difficult for most students to discover on their own. There is a complete proof in this in the instructors’ manual. If you wish to cover this, you will probably want to discuss the proof in class rather than simply assigning it as an exercise.

CHAPTER 14

This final chapter will not satisfy those who are looking for a rigorous treatment of differential geometry. But that is not the point. Instead the objective is to give some intuitive insight into the geometry of the hyperbolic plane and the geometry of the universe. Topics from differential geometry are mentioned only as they are needed.

I have found that students very much enjoy constructing the paper models and find them to be useful in understanding hyperbolic geometry. They also enjoy discussing the geometry of the universe and speculating about it. Given the unrelenting emphasis on the axiomatic method in most of the book, it seems healthy to end with this contrasting view of mathematics. It helps students to see that there are different ways to study and understand mathematics.

There is no point in covering the material on polygonal models unless the students actually make the models themselves and work directly with them. The construction projects make excellent in-class projects. A reasonable thing to do would be to have the students make some of the models in class and then work on other construction projects as (group) homework projects.

The chapter does not have any specific prerequisites, but the questions it addresses will only make sense if you have covered at least some hyperbolic geometry.
Solutions to Exercises in Chapter 1

1.1 Check that the formula \( A = \frac{1}{4}(a + c)(b + d) \) works for rectangles but not for parallelograms.

![Figure S1.1: Exercise 1.1. A rectangle and a parallelogram](image)

For rectangles and parallelograms, \( a = c \) and \( b = d \) and \( Area = base \ast height \). For a rectangle, the base and the height will be equal to the lengths of two adjacent sides. Therefore \( A = a \ast d = \frac{1}{2}(a + a) \ast \frac{1}{2}(b + b) = \frac{1}{4}(a + c)(b + d) \).

In the case of a parallelogram, the height is not always equal to the length of one of the sides so the formula does not work.

1.2 The area of a circle is given by the formula \( A = \pi \left( \frac{d}{2} \right)^2 \). According to the Egyptians, \( A \) is also equal to the area of a square with sides equal to \( \frac{8}{9}d \); thus \( A = (\frac{8}{9})^2d^2 \). Equating and solving for \( \pi \) gives

\[
\pi = \frac{(\frac{8}{9})^2d^2}{\frac{1}{4}d^2} = \frac{64}{81} \cdot \frac{1}{4} = \frac{256}{81} \approx 3.160494.
\]

1.3 The sum of the measures of the two acute angles in \( \triangle ABC \) is 90°, so the first shaded region is a square. We must show that the area of the shaded region in the first square \( (c^2) \) is equal to the area of the shaded region in the second square \( (a^2 + b^2) \).

The two large squares have the same area because they both have side length \( a + b \). Also each of these squares contains four copies of triangle \( \triangle ABC \) (in white). Therefore, by subtraction, the shaded regions must have equal area and so \( a^2 + b^2 = c^2 \).

1.4 (a) Verify that \( (a, b, c) \) is a Pythagorean triple. Suppose \( a = u^2 - v^2, b = 2uv \) and \( c = u^2 + v^2 \). We must show that \( a^2 + b^2 = c^2 \). First, \( a^2 + b^2 = (u^2 - v^2)^2 + (2uv)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2 = u^4 + 2u^2v^2 + v^4 \)

and, second, \( c^2 = (u^2 + v^2)^2 = u^4 + 2u^2v^2 + v^4 = u^4 + 2u^2v^2 + v^4 \).

So \( a^2 + b^2 = c^2 \).
(b) Verify that $a$, $b$, and $c$ are all even if $u$ and $v$ are both odd. Let $u$ and $v$ be odd. We must show that $a$, $b$, and $c$ are even. Since $u$ and $v$ are both odd, we know that $u^2$ and $v^2$ are also odd. Therefore $a = u^2 - v^2$ is even (the difference between two odd numbers is even). It is obvious that $b = 2uv$ is even, and $c = u^2 + v^2$ is also even since it is the sum of two odd numbers.

(c) We must show that $a$, $b$, and $c$ do not have any common factors. Suppose one of $u$ and $v$ is even and the other is odd. Then $a$ and $c$ are both odd, so 2 is not a factor of $a$ or $c$. Suppose $x \neq 2$ is a factor of $b$. Then either $x$ divides $u$ or $x$ divides $v$, but not both because $u$ and $v$ are relatively prime. If $x$ divides $u$, then it also divides $u^2$ but not $v^2$. Thus $x$ is not a factor of $a$ or $c$. If $x$ divides $v$, then it divides $v^2$ but not $u^2$. Again $x$ is not a factor of $a$ or $c$. Therefore $(a, b, c)$ is a primitive Pythagorean triple.

1.5 Use high school geometry to verify that the formula below is correct for a truncated pyramid.

$$V = \frac{h}{3}(a^2 + ab + b^2).$$

Let $h + x$ be the height of the whole pyramid. We know that

$$\frac{h + x}{h} = \frac{a}{b}$$

(by the Similar Triangles Theorem), so $x = h \frac{b}{a - b}$ (algebra). The volume of the truncated pyramid is the volume of the whole pyramid minus the volume of the top pyramid. Therefore

$$V = \frac{1}{3}(h + x)a^2 - \frac{1}{3}xb^2$$

$$= \frac{1}{3}(h + h \frac{b}{a - b})a^2 - \frac{1}{3}h\frac{b^3}{a - b}$$

$$= \frac{h}{3}(a^2 + \frac{a^2b}{a - b}) - \frac{h}{3}\frac{b^3}{a - b}$$

$$= \frac{h}{3}(a^2 + \frac{a^2b - b^3}{a - b})$$

$$= \frac{h}{3}(a^2 + \frac{(a - b)(ab + b^2)}{a - b})$$

$$= \frac{h}{3}(a^2 + ab + b^2).$$

1.6 Constructions using a compass and a straightedge. There is more than one way to accomplish each of these constructions.

(a) The perpendicular bisector of a line segment $\overline{AB}$.

Using the compass, construct two circles, the first about $A$ through $B$, the
second about $B$ through $A$. Then use the straightedge to construct a line through the two points created by the intersection of the two circles.

(b) A line through a point $P$ perpendicular to a line $\ell$.
Use the compass to construct a circle about $P$, making sure the circle is big enough so that it intersects $\ell$ at two points, $A$ and $B$. Then construct the perpendicular bisector of segment $AB$ as in part (a).

(c) The angle bisector of $\angle BAC$.
Using the compass, construct a circle about $A$ that intersects $AB$ and $AC$. Call those points of intersection $D$ and $E$ respectively. Then construct the perpendicular bisector of $DE$. This line is the angle bisector.

1.7 (a) No. Euclid’s postulates say nothing about the number of points on a line.
(b) No.
(c) No. The postulates only assert that there is a line; they do not say there is only one.

1.8 Let \(\square ABCD\) be a rhombus (all four sides are equal), and let \(E\) be the point of intersection between \(\overline{AC}\) and \(\overline{DB}\).\(^1\) We must show that \(\triangle AEB \cong \triangle CEB \cong \triangle CED \cong \triangle AED\). Now \(\angle BAC \equiv \angle ACB\) and \(\angle CAD \equiv \angle ACD\) by Proposition 5. By addition we can see that \(\angle BAD \equiv \angle BCD\) and similarly, \(\angle ADC \equiv \angle ABC\). Now we know that \(\triangle ABC \cong \triangle ADC\) by Proposition 4. Similarly, \(\triangle DBA \cong \triangle DBC\). This implies that \(\angle BAC \equiv \angle DAC \equiv \angle BCA \equiv \angle DCA\) and \(\angle BDA \equiv \angle BDC \equiv \angle DBC\). Thus \(\triangle AEB \cong \triangle AED \cong \triangle CEB \cong \triangle CED\), again by Proposition 4.\(^2\)

---

\(^1\)In this solution and the next, the existence of the point \(E\) is taken for granted. Its existence is obvious from the diagram. Proving that \(E\) exists is one of the gaps that must be filled in these proofs. This point will be addressed in Chapter 6.

\(^2\)It should be noted that the fact about rhombi can be proved using just propositions that
1.9 Let \( \square ABCD \) be a rectangle, and let \( E \) be the point of intersection of \( \overline{AC} \) and \( \overline{BD} \). We must prove that \( \overline{AC} \equiv \overline{BD} \) and that \( \overline{AC} \) and \( \overline{BD} \) bisect each other (i.e., \( \overline{AE} \equiv \overline{EC} \) and \( \overline{BE} \equiv \overline{ED} \)). By Proposition 28, \( \overrightarrow{DA} \parallel \overrightarrow{CB} \) and \( \overrightarrow{DC} \parallel \overrightarrow{AB} \). Therefore, by Proposition 29, \( \angle CAB \equiv \angle ACD \) and \( \angle DAC \equiv \angle ACB \). Hence \( \triangle ABC \equiv \triangle CDA \) and \( \triangle ADB \equiv \triangle CBD \) by Proposition 26 (ASA). Since those triangles are congruent we know that opposite sides of the rectangle are congruent and \( \triangle ABD \equiv \triangle BAC \) (by Proposition 4), and therefore \( \overline{BD} \equiv \overline{AC} \).

Now we must prove that the segments bisect each other. By Proposition 29, \( \angle CAB \equiv \angle ACD \) and \( \angle DBA \equiv \angle BDC \). Hence \( \triangle ABE \equiv \triangle CDE \) (by Proposition 26) which implies that \( \overline{AE} \equiv \overline{CE} \) and \( \overline{DE} \equiv \overline{BE} \). Therefore the diagonals are equal and bisect each other.

1.10 The argument works for the first case. This is the case in which the triangle actually is isosceles. The second case never occurs (\( D \) is never inside the triangle). The flaw lies in the third case (\( D \) is outside the triangle). If the triangle is not isosceles then either \( E \) will be outside the triangle and \( F \) will be on the edge \( \overline{AC} \), or \( E \) will be on the edge \( \overline{AB} \) and \( F \) will be outside. They cannot both be outside as shown in the diagram. This can be checked by drawing a careful diagram by hand or by drawing the diagram using Geometer’s Sketchpad (or similar software).

come early in Book I and do not depend on the Fifth Postulate, whereas the proof in the next exercise requires propositions about parallelism that Euclid proves much later in Book I using his Fifth Postulate.
Solutions to Exercises in Chapter 2

2.1 This is not a model for Incidence Geometry since it does not satisfy Incidence Axiom 3. This example is isomorphic to the 3-point line.

2.2 One-point Geometry satisfies Axioms 1 and 2 but not Axiom 3. Every pair of distinct points defines a unique line (vacuously—there is no pair of distinct points). There do not exist three distinct points, so there cannot be three noncollinear points. One-point Geometry satisfies all three parallel postulates (vacuously—there is no line).

2.3 It helps to draw a schematic diagram of the relationships.

![Figure S2.1: A schematic representation of the committee structures](image)

(a) Not a model. There is no line containing B and D. There are two lines containing B and C.
(b) Not a model. There is no line containing C and D.
(c) Not a model. There is no line containing A and D.

2.4 (a) The Three-point plane is a model for Three-point geometry.
(b) Every model for Three-point geometry has 3 lines. If there are 3 points, then there are also 3 pairs of points
(c) Suppose there are two models for Three-point geometry, model A and model B. Choose any 1-1 correspondence of the points in model A to the points in model B. Any line in A is determined by two points. These two points correspond to two points in B. Those two points determine a line in B. The isomorphism should map the given line in A to this line in B. Then the function will preserve betweenness. Therefore models A and B are isomorphic to one another.

2.5 Axiom 1 does not hold, but Axioms 2 and 3 do. The Euclidean Parallel Postulate holds. The other parallel postulates are false in this interpretation.

2.6 See Figure S2.2
2.7 Fano’s Geometry satisfies the Elliptic Parallel Postulate because every line shares at least one point with every other line; there are no parallel lines. It does not satisfy either of the other parallel postulates.

2.8 The three-point line satisfies all three parallel postulates (vacuously).

2.9 If there are so few points and lines that there is no line with an external point, then all three parallel postulates are satisfied (vacuously). If there is a line with an external point, then there will either be a parallel line through the external point or there will not be. Hence at most one of the parallel postulates can be satisfied in that case. Since every incidence geometry contains three noncollinear points, there must be a line with an external point. Hence an incidence geometry can satisfy at most one of the parallel postulates.

2.10 Start with a line with three points on it. There must exist another point not on that line (Incidence Axiom 3). That point, together with the points on the original line, determines three more lines (Incidence Axiom 1). But each of those lines must have a third point on it. So there must be at least three more points, for a total of at least seven points. Since Fano’s Geometry has exactly seven points, seven is the minimum.

2.11 See Figures S2.3 and S2.4.

2.12 (a) The three-point line (Example 2.6.2).
(b) The square (Exercise 2.5) or the sphere (Example 2.6.7).
(c) One-point geometry (Exercise 2.2).

2.13 (a) The dual of the Three-point plane is another Three-point plane. It is a model for incidence geometry.
(b) The dual of the Three-point line is a point which is incident with 3 lines. This is not a model for incidence geometry.
(c) The dual of Four-point Geometry has 6 points and 4 lines. Each point is incident with exactly 2 lines, and each line is incident with 3 points. It is not a model for incidence geometry because it does not satisfy Incidence
Axiom 1.

(d) The dual of Fano’s Geometry is isomorphic to Fano’s Geometry, so it is a model for incidence geometry.
Solutions to Exercises in Chapter 3

3.1  (a) ∀ model for incidence geometry, the Euclidean Parallel Postulate does not hold in that model.
     (b) ∃ a model for incidence geometry in which there are not exactly 7 points (the number of points is either ≤ 6 or ≥ 8).
     (c) ∃ a triangle whose angle sum is not 180°.
     (d) It is not hot or it is not humid outside.
     (e) My favorite color is not red and it is not green.
     (f) The sun shines and (but?) we do not go hiking. (See explanation in last full paragraph on page 36.)
     (g) ∃ a geometry student who does not know how to write proofs.

3.2  (a) Negation of Euclidean Parallel Postulate. There exist a line ℓ and a point P not on ℓ such that either there is no line m such that P lies on m and m is parallel to ℓ or there are (at least) two lines m and n such that P lies on both m and n, m || ℓ, and n || ℓ.
     (b) Negation of Elliptic Parallel Postulate. There exist a line ℓ and a point P that does not lie on ℓ such that there is at least one line m such that P lies on m and m || ℓ.
     (c) Negation of Hyperbolic Parallel Postulate. There exist a line ℓ and a point P that does not lie on ℓ such that either there is no line m such that P lies on m and m || ℓ or there is exactly one line m with these properties.

Note. You could emphasize the separate existence of ℓ and P by starting each of the statements above with, “There exist a line ℓ and there exists a point P not on ℓ such that ....”

3.3  \( \text{not (} S \text{ and } T \text{)} \equiv (\text{not } S) \text{ or (not } T \text{)} \)

<table>
<thead>
<tr>
<th>S</th>
<th>T</th>
<th>S and T</th>
<th>not (S and T)</th>
<th>not S</th>
<th>not T</th>
<th>(not C) or (not H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>True</td>
<td>True</td>
<td>False</td>
<td>False</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>True</td>
<td>False</td>
<td>False</td>
<td>True</td>
<td>False</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>False</td>
<td>True</td>
<td>False</td>
<td>True</td>
<td>True</td>
<td>False</td>
<td>True</td>
</tr>
<tr>
<td>False</td>
<td>False</td>
<td>True</td>
<td>True</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
</tbody>
</table>