INSTRUCTOR'S SOLUTIONS MANUAL

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FUNDAMENTALS OF DIFFERENTIAL EQUATIONS

EIGHTH EDITION

and

FUNDAMENTALS OF DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

SIXTH EDITION

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Notes to the Instructor

One goal in our writing has been to create flexible texts that afford the instructor a variety of topics and make available to the student an abundance of practice problems and projects. We recommend that the instructor read the discussion given in the preface in order to gain an overview of the prerequisites, topics of emphasis, and general philosophy of the text.

Supplements

Student's Solutions Manual: By Viktor Maymeskul. Contains complete, worked-out solutions to most odd-numbered exercises, providing students with an excellent study tool. ISBN 13: 978-0-321-74834-8; ISBN 10: 0-321-74834-4.

Companion Web site: Provides additional resources for both instructors and students, including helpful links keyed to sections of the text, access to Interactive Differential Equations, suggestions for incorporating Interactive Differential Equations modules, suggested syllabi, index of applications, and study tips for students. Access: `www.pearsonhighered.com/nagle`

Interactive Differential Equations: By Beverly West (Cornell University), Steven Strogatz (Cornell University), Jean Marie McDill (California Polytechnic State University – San Luis Obispo), John Cantwell (St. Louis University), and Hubert Hohn (Massachusetts College of Arts) is a popular software directly tied to the text that focuses on helping students visualize concepts. Applications are drawn from engineering, physics, chemistry, and biology. Access: `www.pearsonhighered.com/nagle`

Instructor's MAPLE/MATHLAB/MATHEMATICA manuals: By Thomas W. Polaski (Winthrop University), Bruno Welfert (Arizona State University), and Maurino Bautista (Rochester Institute of Technology). A collection of worksheets and projects to aid instructors in integrating computer algebra systems into their courses. Available in the Pearson Instructor Resource Center at `www.pearsonhighered.com/irc`.

MATLAB Manual ISBN 13: 978-0-321-53015-8; ISBN 10: 0-321-53015-2
MAPLE Manual ISBN 13: 978-0-321-38842-1; ISBN 10: 0-321-38842-9
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Computer Labs

A computer lab in connection with a differential equations course can add a whole new dimension to the teaching and learning of differential equations. As more and more colleges and universities set up computer labs with software such as MAPLE, MATLAB, DERIVE, MATHEMATICA, PHASEPLANE, and MACMATH, there will be more opportunities to include a lab as part of the differential equations course. In our teaching and in our texts, we have tried to provide a variety of exercises, problems, and projects that encourage the student to use the computer to explore. Even one or two hours at a computer generating phase plane diagrams can provide the students with a feeling of how they will use technology together with the theory to investigate real world problems. Furthermore, our experience is that they thoroughly enjoy these activities. Of course, the software, provided free with the texts, is especially convenient for such labs.

Group Projects

Although the projects that appear at the end of the chapters in the text can be worked out by the conscientious student working alone, making them *group* projects adds a social element that encourages discussion and interactions that simulate a professional work place atmosphere. Group sizes of 3 or 4 seem to be optimal. Moreover, requiring that each individual student separately write up the group's solution as a formal technical report for grading by the instructor also contributes to the professional flavor.

Typically, our students each work on 3 or 4 projects per semester. If class time permits, oral presentations by the groups can be scheduled and help to improve the communication skills of the students.

The role of the instructor is, of course, to help the students solve these elaborate problems on their own and to recommend additional reference material when appropriate.

Some additional Group Projects are presented in this guide (see page 10).

Technical Writing Exercises

The technical writing exercises at the end of most chapters invite students to make documented responses to questions dealing with the concepts in the chapter. This not only gives students an opportunity to improve their writing skills, but it helps them organize their thoughts and better understand the new concepts. Moreover, many questions deal with critical thinking

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skills that will be useful in their careers as engineers, scientists, or mathematicians.

Since most students have little experience with technical writing, it may be necessary to return *ungraded* the first few technical writing assignments with comments and have the students redo the the exercise. This has worked well in our classes and is much appreciated by the students. Handing out a "model" technical writing response is also helpful for the students.

Student Presentations

It is not uncommon for an instructor to have students go to the board and present a solution to a problem. Differential equations is so rich in theory and applications that it is an excellent course to allow (require) a student to give a presentation on a special application (e.g., almost any topic from Chapters 3 and 5), on a new technique not covered in class (e.g., material from Section 2.6, Projects A, B, or C in Chapter 4), or on additional theory (e.g., material from Chapter 6 which generalizes the results in Chapter 4). In addition to improving students' communication skills, these "special" topics are long remembered by the students. Here, too, working in groups of 3 or 4 and sharing the presentation responsibilities can add substantially to the interest and quality of the presentation. Students should also be encouraged to enliven their communication by building physical models, preparing part of their lectures with the aid of video technology, and utilizing appropriate internet web sites.

Homework Assignments

We would like to share with you an obvious, non-original, but effective method to encourage students to do homework problems.

An essential feature is that it requires little extra work on the part of the instructor or grader. We assign homework problems (about 5 of them) after each lecture. At the end of the week (Fridays), students are asked to turn in their homework (typically, 3 sets) for that week. We then choose at random one problem from each assignment (typically, a total of 3) that will be graded. (The point is that the student does not know in advance which problems will be chosen.) Full credit is given for any of the chosen problems for which there is evidence that the student has made an honest attempt at solving. The homework problem sets are returned to the students at the next meeting (Mondays) with grades like 0/3, 1/3, 2/3, or 3/3 indicating the proportion of problems for which the student received credit. The homework grades are tallied at the end of the semester and count as one test grade. Certainly, there are variations

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on this theme. The point is that students are motivated to do their homework.

Syllabus Suggestions

To serve as a guide in constructing a syllabus for a one-semester or two-semester course, the prefaces to the texts list sample outlines that emphasize methods, applications, theory, partial differential equations, phase plane analysis, computation, or combinations of these. As a further guide in making a choice of subject matter, we provide (starting on the next page) a listing of text material dealing with some common areas of emphasis.

Numerical, Graphical, and Qualitative Methods

The sections and projects dealing with numerical, graphical, and qualitative techniques of solving differential equations include:

Section 1.3: *Direction Fields*

Section 1.4: *The Approximation Method of Euler*

Project A for Chapter 1: *Taylor Series Method*

Project B for Chapter 1: *Picard's Method*

Project C for Chapter 1: *The Phase Line*

Section 3.6: *Improved Euler's Method*, which includes step-by-step outlines of the improved Euler's method subroutine and improved Euler's method with tolerance. These outlines are easy for the student to translate into a computer program (pp. 127–128).

Section 3.7: *Higher-Order Numerical Methods*: *Taylor and Runge-Kutta*, which includes outlines for the Fourth Order Runge-Kutta subroutine and algorithm with tolerance (see pp. 135–136).

Project H for Chapter 3: *Stability of Numerical Methods*

Project I for Chapter 3: *Period Doubling and Chaos*

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Section 4.8: *Qualitative Considerations for Variable Coefficient and Non-linear Equations*, which discusses the energy integral lemma, as well as the Airy, Bessel, Duffing, and van der Pol equations.

Section 5.3: *Solving Systems and Higher-Order Equations Numerically*, which describes the vectorized forms of Euler's method and the Fourth Order Runge-Kutta method, and discusses an application to population dynamics.

Section 5.4: *Introduction to the Phase Plane*, which introduces the study of trajectories of autonomous systems, critical points, and stability.

Section 5.8: *Dynamical Systems, Poincaré Maps, and Chaos*, which discusses the use of numerical methods to approximate the Poincarè map and how to interpret the results.

Project A for Chapter 6: *Computer Algebra Systems and Exponential Shift*

Project D for Chapter 6: *Higher-Order Difference Equations*

Project A for Chapter 8: *Alphabetization Algorithms*

Project D for Chapter 10: *Numerical Method for $\Delta u = f$ on a Rectangle*

Project D for Chapter 11: *Shooting Method*

Project E for Chapter 11: *Finite-Difference Method for Boundary Value Problems*

Project C for Chapter 12: *Computing Phase Plane Diagrams*

Project D for Chapter 12: *Ecosystem of Planet GLIA-2*

Section 13.1: *Introduction: Successive Approximations*

Appendix B: *Newton's Method*

Appendix C: *Simpson's Rule*

Appendix E: *Method of Least Squares*

Appendix F: *Runge-Kutta Procedure for Equations*

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The instructor who wishes to emphasize numerical methods should also note that the text contains an extensive chapter of series solutions of differential equations (Chapter 8).

Engineering/Physics Applications

Since Laplace transforms is a subject vital to engineering, we have included a detailed chapter on this topic – see Chapter 7. Stability is also an important subject for engineers, so we have included an introduction to the subject in Section 5.4 along with an entire chapter addressing this topic – see Chapter 12. Further material dealing with engineering/physics applications include:

Project C for Chapter 2: *Torricelli's Law of Fluid Flow.*

Project I for Chapter 2: *Designing a Solar Collector.*

Section 3.1: *Mathematical Modeling.*

Section 3.2: *Compartmental Analysis*, which contains a discussion of mixing problems and of population models.

Section 3.3: *Heating and Cooling off Buildings*, which discusses temperature variations in the presence of air conditioning or furnace heating.

Section 3.4: *Newtonian Mechanics.*

Section 3.5: *Electrical Circuits.*

Project C for Chapter 3: *Curve of Pursuit.*

Project D for Chapter 3: *Aircraft Guidance in a Crosswind.*

Project E for Chapter 3: *Feedback and the Op Amp.*

Project F for Chapter 3: *Bang-Bang Controls.*

Section 4.1: *Introduction: The Mass-Spring Oscillator.*

Section 4.8: *Qualitative Considerations for Variable-Coefficient and Non-linear Equations.*

Section 4.9: *A Closer Look at Free Mechanical Vibrations.*

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Section 4.10: *A Closer Look at Forced Mechanical Vibrations.*

Project B for Chapter 4: *Apollo Re-entry*

Project C for Chapter 4: *Simple Pendulum*

Section 5.1: *Interconnected Fluid Tanks.*

Section 5.4: *Introduction to the Phase PLane.*

Section 5.6: *Coupled Mass-Spring Systems.*

Section 5.7: *Electrical Systems.*

Section 5.8: *Dynamical Systems, Poincaré Maps, and Chaos .*

Project A for Chapter 5: *Designing a Landing System for Interplanetary Travel.*

Project C for Chapter 5: *Things that Bob.*

Project D for Chapter 5: *Hamiltonian Systems.*

Project C for Chapter 6: *Transverse Vibrations of a Beam.*

Chapter 7: *Laplace Transforms*, which in addition to basic material includes discussions of transfer functions, the Dirac delta function, and frequency response modelling.

Project B for Chapter 8, *Spherically Symmetric Solutions to Schrödinger's Equation for the Hydrogen Atom*

Project D for Chapter 8, *Buckling of a Tower*

Project E for Chapter 8, *Aging Spring and Bessel Functions*

Section 9.6: *Complex Eigenvalues*, includes discussion of normal (natural) frequencies.

Project B for Chapter 9: *Matrix Laplace Transform Method.*

Project C for Chapter 9: *Undamped Second-Order Systems.*

Chapter 10: *Partial Differential Equations*, which includes sections on Fourier series, the heat equation, wave equation, and Laplace's equation.

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Project A for Chapter 10: *Steady-State Temperature Distribution in a Circular Cylinder*.

Project B for Chapter 10: *A Laplace Transform Solution of the Wave Equation*.

Project A for Chapter 11: *Hermite Polynomials and the Harmonic Oscillator*.

Section 12.4: *Energy Methods*, which addresses both conservative and non-conservative autonomous mechanical systems.

Project A for Chapter 12: *Solitons and Korteweg-de Vries Equation*.

Project B for Chapter 12: *Burger's Equation*.

Students of engineering and physics would also find Chapter 8 on series solutions particularly useful, especially Section 8.8 on special functions.

Biology/Ecology Applications

Project C for Chapter 1: *The Phase Plane*, which discusses the logistic population model and bifurcation diagrams for population control.

Project A for Chapter 2: *Oil Spill in a Canal*.

Project B for Chapter 2: *Differential Equations in Clinical Medicine*.

Section 3.1: *Mathematical Modelling*.

Section 3.2: *Compartmental Analysis*, which contains a discussion of mixing problems and population models.

Project A for Chapter 3: *Dynamics of HIV Infection*.

Project B for Chapter 3: *Aquaculture*, which deals with a model of raising and harvesting catfish.

Section 5.1: *Interconnected Fluid Tanks*, which introduces systems of equations.

Section 5.3: *Solving Systems and Higher-Order Equations Numerically*, which contains an application to population dynamics.

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Section 5.5: *Applications to Biomathematics: Epidemic and Tumor Growth Models*.

Project B for Chapter 5: *Spread of Staph Infections in Hospitals – Part I.*

Project E for Chapter 5: *Cleaning Up the Great Lakes*

Project F for Chapter 5: *A Growth Model for Phytoplankton – Part I.*

Problem 19 in Exercises 10.5 , which involves chemical diffusion through a thin layer.

Project D for Chapter 12: *Ecosystem on Planet GLIA-2*

Project E for Chapter 12: *Spread of Staph Infections in Hospitals – Part II.*

Project F for Chapter 12: *A Growth Model for Phytoplankton – Part II.*

The basic content of the remainder of this instructor's manual consists of supplemental group projects, answers to the even-numbered problems, and detailed solutions to the most even-numbered problems in Chapters 1, 2, 3, 4, and 7. These answers are not available any place else since the text and the *Student's Solutions Manual* only provide answers and solutions to odd-numbered problems.

We would appreciate any comments you may have concerning the answers in this manual. These comments can be sent to the authors' email addresses below. We also would encourage sharing with us (the authors and users of the texts) any of your favorite group projects.

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Group Projects for Chapter 3

Delay Differential Equations

In our discussion of mixing problems in Section 3.2, we encountered the initial value problem

$$x'(t) = 6 - \frac{3}{500}\, x\,(t - t_0)\,, \qquad (0.1)$$
$$x(t) = 0 \quad \text{for} \quad x \in [-t_0, 0]\,,$$

where t_0 is a positive constant. The equation in (0.1) is an example of a **delay differential equation**. These equations differ from the usual differential equations by the presence of the shift $(t - t_0)$ in the argument of the unknown function $x(t)$. In general, these equations are more difficult to work with than are regular differential equations, but quite a bit is known about them.[1]

(a) Show that the simple linear delay differential equation

$$x' = ax(t - b), \qquad (0.2)$$

where a, b are constants, has a solution of the form $x(t) = Ce^{st}$ for any constant C, provided s satisfies the transcendental equation $s = ae^{-bs}$.

(b) A solution to (0.2) for $t > 0$ can also be found using the **method of steps**. Assume that $x(t) = f(t)$ for $-b \le t \le 0$. For $0 \le t \le b$, equation (0.2) becomes

$$x'(t) = ax(t - b) = af(t - b),$$

and so

$$x(t) = \int_0^t af(\nu - b)d\nu + x(0).$$

Now that we know $x(t)$ on $[0, b]$, we can repeat this procedure to obtain

$$x(t) = \int_b^t ax(\nu - b)d\nu + x(b)$$

for $b \le x \le 2b$. This process can be continued indefinitely.

[1] See, for example, *Differential–Difference Equations*, by R. Bellman and K. L. Cooke, Academic Press, New York, 1963, or *Ordinary and Delay Differential Equations*, by R. D. Driver, Springer–Verlag, New York, 1977

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Use the method of steps to show that the solution to the initial value problem

$$x'(t) = -x(t-1), \qquad x(t) = 1 \quad \text{on} \quad [-1, 0],$$

is given by

$$x(t) = \sum_{k=0}^{n} (-1)^k \frac{[t-(k-1)]^k}{k!}, \qquad \text{for} \quad n-1 \le t \le n,$$

where n is a nonnegative integer. (This problem can also be solved using the Laplace transform method of Chapter 7.)

(c) Use the method of steps to compute the solution to the initial value problem given in (0.1) on the interval $0 \le t \le 15$ for $t_0 = 3$.

Extrapolation

When precise information about the *form* of the error in an approximation is known, a technique called **extrapolation** can be used to improve the rate of convergence.

Suppose the approximation method converges with rate $O(h^p)$ as $h \to 0$ (cf. Section 3.6). From theoretical considerations, assume we know, more precisely, that

$$y(x;h) = \phi(x) + h^p a_p(x) + O\left(h^{p+1}\right), \tag{0.3}$$

where $y(x;h)$ is the approximation to $\phi(x)$ using step size h and $a_p(x)$ is some function that is independent of h (typically, we do not know a formula for $a_p(x)$, only that it exists). Our goal is to obtain approximations that converge at the faster rate than $O(h^{p+1})$.

We start by replacing h by $h/2$ in (0.3) to get

$$y\left(x; \frac{h}{2}\right) = \phi(x) + \frac{h^p}{2^p} a_p(x) + O\left(h^{p+1}\right).$$

If we multiply both sides by 2^p and subtract equation (0.3), we find

$$2^p y\left(x; \frac{h}{2}\right) - y(x;h) = (2^p - 1)\,\phi(x) + O\left(h^{p+1}\right).$$

Solving for $\phi(x)$ yields

$$\phi(x) = \frac{2^p y(x; h/2) - y(x;h)}{2^p - 1} + O\left(h^{p+1}\right).$$

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Hence,

$$y^*\left(x;\frac{h}{2}\right) := \frac{2^p y\left(x;h/2\right) - y(x;h)}{2^p - 1}$$

has a rate of convergence of $\mathrm{O}\left(h^{p+1}\right)$.

(a) Assuming

$$y^*\left(x;\frac{h}{2}\right) = \phi(x) + h^{p+1}a_{p+1}(x) + \mathrm{O}\left(h^{p+2}\right),$$

show that

$$y^{**}\left(x;\frac{h}{4}\right) := \frac{2^{p+1} y^*\left(x;h/4\right) - y^*(x;h/2)}{2^{p+1} - 1}$$

has a rate of convergence of $\mathrm{O}\left(h^{p+2}\right)$.

(b) Assuming

$$y^{**}\left(x;\frac{h}{4}\right) = \phi(x) + h^{p+2}a_{p+2}(x) + \mathrm{O}\left(h^{p+3}\right),$$

show that

$$y^{***}\left(x;\frac{h}{8}\right) := \frac{2^{p+2} y^{**}\left(x;h/8\right) - y^{**}(x;h/4)}{2^{p+2} - 1}$$

has a rate of convergence of $\mathrm{O}\left(h^{p+3}\right)$.

(c) The results of using Euler's method (with $h = 1,\ 1/2,\ 1/4,\ 1/8$) to approximate the solution to the initial value problem

$$y' = y, \quad y(0) = 1$$

at $x = 1$ are given in Table 1.2, page 26. For Euler's method, the extrapolation procedure applies with $p = 1$. Use the results in Table 1.2 to find an approximation to $e = y(1)$ by computing $y^{***}(1;1/8)$. [Hint: Compute $y^*\left(1;1/2\right)$, $y^*\left(1;1/4\right)$, and $y^*\left(1;1/8\right)$; then compute $y^{**}\left(1;1/4\right)$ and $y^{**}\left(1;1/8\right)$.]

(d) Table 1.2 also contains Euler's approximation for $y(1)$ when $h = 1/16$. Use this additional information to compute the next step in the extrapolation procedure; that is, compute $y^{****}(1;1/16)$.

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Group Projects for Chapter 5

Effects of Hunting on Predator–Prey Systems

As discussed in Section 5.3 (pp. 257–259), cyclic variations in the population of predators and their prey have been studied using the Volterra-Lotka predator–prey model

$$\frac{dx}{dt} = Ax - Bxy\,, \tag{0.4}$$

$$\frac{dy}{dt} = -Cy + Dxy\,, \tag{0.5}$$

where A, B, C, and D are positive constants, $x(t)$ is the population of prey at time t, and $y(t)$ is the population of predators. It can be shown that such a system has a periodic solution. That is, there exists some constant T such that $x(t) = x(t+T)$ and $y(t) = y(t+T)$ for all t. The periodic or cyclic variation in the population has been observed in various systems such as sharks–food fish, lynx–rabbits, and ladybird beetles–cottony cushion scale. Because of this periodic behavior, it is useful to consider the average population $\overline{x}$ and $\overline{y}$ defined by

$$\overline{x} := \frac{1}{T}\int_0^t x(t)dt\,, \qquad \overline{y} := \frac{1}{T}\int_0^t y(t)dt\,.$$

(a) Show that $\overline{x} = C/D$ and $\overline{y} = A/B$. [Hint: Use equation (0.4) and the fact that $x(0) = x(T)$ to show that

$$\int_0^T [A - By(t)]\,dt = \int_0^T \frac{x'(t)}{x(t)}\,dt = 0\,.$$

(b) To determine the effect of indiscriminate hunting on the population, assume hunting reduces the rate of change in a population by a constant times the population. Then the predator–prey system satisfies the new set of equations

$$\frac{dx}{dt} = Ax - Bxy - \varepsilon x = (A-\varepsilon)x - Bxy\,, \tag{0.6}$$

$$\frac{dy}{dt} = -Cy + Dxy - \delta y = -(C+\delta)y + Dxy\,, \tag{0.7}$$

where ε and δ are positive constants with $\varepsilon < A$. What effect does this have on the average population of prey? On the average population of predators?

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(c) Assume the hunting was done selectively, as in shooting only rabbits (or shooting only lynx). Then we have $\varepsilon > 0$ and $\delta = 0$ (or $\varepsilon = 0$ and $\delta > 0$) in (0.6)–(0.7). What effect does this have on the average populations of predator and prey?

(d) In a rural county, foxes prey mainly on rabbits but occasionally include a chicken in their diet. The farmers decide to put a stop to the chicken killing by hunting the foxes. What do you predict will happen? What will happen to the farmers' gardens?

Limit Cycles

In the study of triode vacuum tubes, one encounters the van der Pol equation[2]

$$y'' - \mu\left(1 - y^2\right)y' + y = 0\,,$$

where the constant μ is regarded as a parameter. In Section 4.8, we used the mass-spring oscillator analogy to argue that the non-zero solutions to the van der Pol equation with $\mu = 1$ should approach a periodic limit cycle. The same argument applies for any positive value of μ.

(a) Recast the van der Pol equation as a system in normal form and use software to plot some typical trajectories for $\mu = 0.1$, 1, and 10. Re-scale the plots if necessary until you can discern the limit cycle trajectory; find trajectories that spiral in, and ones that spiral out, to the limit cycle.

(b) Now let $\mu = -0.1$, -1, and -10. Try to predict the nature of the solutions using the mass-spring analogy. Then use the software to check your predictions. Are there limit cycles? Do the neighboring trajectories spiral into, or spiral out from, the limit cycles?

(c) Repeat parts (a) and (b) for the Rayleigh equation

$$y'' - \mu\left[1 - (y')^2\right]y' + y = 0\,.$$

[2]*Historical Footnote*: Experimental research by **E. V. Appleton** and **B. van der Pol** in 1921 on the oscillation of an electrical circuit containing a triode generator (vacuum tube) led to the non-linear equation now called 0 **van der Pol's equation**. Methods of solution were developed by van der Pol in 1926–1927. **Mary L. Cartwright** continued research into non-linear oscillation theory and together with **J. E. Littlewood** obtained existence results for forced oscillations in non-linear systems in 1945.

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Group Project for Chapter 13

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Satellite Altitude Stability

In this problem, we determine the orientation at which a satellite in a circular orbit of radius r can maintain a relatively constant facing with respect to a spherical primary (e.g., a planet) of mass M. The torque of gravity on the asymmetric satellite maintains the orientation.

Suppose (x, y, z) and $(\overline{x}, \overline{y}, \overline{z})$ refer to coordinates in two systems that have a common origin at the satellite's center of mass. Fix the xyz-axes in the satellite as principal axes; then let the $\overline{z}$-axis point toward the primary and let the $\overline{x}$-axis point in the direction of the satellite's velocity. The xyz-axes may be rotated to coincide with the $\overline{xyz}$-axes by a rotation ϕ about the x-axis (roll), followed by a rotation θ about the resulting y-axis (pitch), and a rotation ψ about the final z-axis (yaw). Euler's equations from physics (with high terms omitted[3] to obtain approximate solutions valid near $(\phi, \theta, \psi) = (0, 0, 0)$) show that the equations for the rotational motion due to gravity acting on the satellite are

$$
\begin{aligned}
I_x\phi'' &= -4\omega_0^2\,(I_z - I_y)\,\phi - \omega_0\,(I_y - I_z - I_x)\,\psi' \\
I_y\theta'' &= -3\omega_0^2\,(I_x - I_z)\,\theta \\
I_z\psi'' &= -4\omega_0^2\,(I_y - I_x)\,\psi + \omega_0\,(I_y - I_z - I_x)\,\phi'\,,
\end{aligned}
$$

where $\omega_0 = \sqrt{(GM)/r^3}$ is the angular frequency of the orbit and the positive constants I_x, I_y, I_z are the moments of inertia of the satellite about the x, y, and z-axes.

(a) Find constants $c_1, \ldots, c_5$ such that these equations can be written as two systems

$$
\frac{d}{dt}\begin{bmatrix} \phi \\ \psi \\ \phi' \\ \theta' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_1 & 0 & 0 & c_2 \\ 0 & c_3 & c_4 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \phi' \\ \psi' \end{bmatrix}
$$

and

$$
\frac{d}{dt}\begin{bmatrix} \theta \\ \theta' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c_5 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \theta' \end{bmatrix}.
$$

[3] The derivation of these equations is found in *Attitude Stabilization and Control of Earth Satellites*, by O. H. Gerlach, Space Science Reviews, #4 (1965), 541–566.

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(b) Show that the origin is asymptotically stable for the first system in (a) if

$$(c_2c_4 + c_3 + c_1)^2 - 4c_1c_3 > 0\,,$$
$$c_1c_3 > 0\,,$$
$$c_2c_4 + c_3 + c_1 > 0$$

and hence deduce that $I_y > I_x > I_z$ yields an asymptotically stable origin. Are there other conditions on the moments of inertia by which the origin is stable?

(c) Show that, for the asymptotically stable configuration in (b), the second system in (a) becomes a harmonic oscillator problem, and find the frequency of oscillation in terms of I_x, I_y, I_z, and ω_0. Phobos maintains $I_y > I_x > I_z$ in its orientation with respect to Mars, and has angular frequency of orbit $\omega_0 = 0.82$ rad/hr. If $(I_x - I_z)/I_y = 0.23$, show that the period of the libration for Phobos (the period with which the side of Phobos facing Mars shakes back and forth) is about 9 hours.

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CHAPTER 1: Introduction

EXERCISES 1.1: Background

2. This equation involves only ordinary derivatives of x with respect to t, and the highest derivative has the second order. Thus it is an ordinary differential equation of the second order with independent variable t and dependent variable x. It is linear because x, dx/dt, and d^2x/dt^2 appear in additive combination (even with constant coefficients) of their first powers.

4. This equation is an ODE because it contains no partial derivatives. Since the highest order derivative is dy/dx, the equation is a first order equation. This same term also shows us that the independent variable is x and the dependent variable is y. This equation is nonlinear because of the y in the denominator of $[y(2-3x)]/[x(1-3y)]$.

6. This equation is an ordinary first order differential equation with independent variable x and dependent variable y. It is nonlinear because of the term containing the square of dy/dx.

8. This equation is an ODE because it contains only ordinary derivatives. The term dp/dt is the highest order derivative and thus shows us that this is a first order equation. This term also shows us that the independent variable is t and the dependent variable is p. This equation is nonlinear since in the term $kp(P-p) = kPp - kp^2$ the dependent variable p is squared (compare with equation (7)).

10. This equation contains only ordinary derivatives of y with respect to x. Hence, it is an ordinary differential equation of the second order (because the highest order derivative is d^2y/dx^2) with independent variable x and dependent variable y. This equation is of the form (7) and, therefore, is linear.

12. ODE of the second order with the independent variable x and the dependent variable y, nonlinear.

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14. The velocity at time t is the rate of change of the position function $x(t)$, i.e., x'. Thus,

$$\frac{dx}{dt} = kx^4,$$

where k is the proportionality constant.

16. The equation is

$$\frac{dA}{dt} = kA^2,$$

where k is the proportionality constant.

EXERCISES 1.2: Solutions and Initial Value Problems

2. (a) Differentiating $\phi(x)$ yields $\phi'(x) = 2x$. Substitution ϕ and ϕ' for y and y' into the given equation, $xy' = 2y$, gives $x(2x) = 2(x^2)$, which is an identity on $(-\infty, \infty)$. Thus, $\phi(x)$ is an explicit solution on $(-\infty, \infty)$.

(b) We compute

$$\frac{d\phi}{dx} = \frac{d}{dx}(e^x - x) = e^x - 1.$$

Functions $\phi(x)$ and $\phi'(x)$ are defined for all real numbers and

$$\begin{aligned}\frac{d\phi}{dx} + \phi(x)^2 &= (e^x - 1) + (e^x - x)^2 \\ &= (e^x - 1) + \left(e^{2x} - 2xe^x + x^2\right) = e^{2x} + (1 - 2x)e^x + x^2 - 1,\end{aligned}$$

which is identically equal to the right-hand side of the given equation. Thus, $\phi(x)$ is an explicit solution on $(-\infty, \infty)$.

(c) Note that the function $\phi(x) = x^2 - x^{-1}$ is not defined at $x = 0$. Differentiating $\phi(x)$ twice yields

$$\begin{aligned}\frac{d\phi}{dx} &= \frac{d}{dx}\left(x^2 - x^{-1}\right) = 2x - (-1)x^{-2} = 2x + x^{-2}; \\ \frac{d^2\phi}{dx^2} &= \frac{d}{dx}\left(\frac{d\phi}{dx}\right) = \frac{d}{dx}\left(2x + x^{-2}\right) = 2 + (-2)x^{-3} = 2\left(1 - x^{-3}\right).\end{aligned}$$

Therefore,

$$x^2\frac{d^2\phi}{dx^2} = x^2 \cdot 2\left(1 - x^{-3}\right) = 2\left(x^2 - x^{-1}\right) = 2\phi(x),$$

and $\phi(x)$ is an explicit solution to the differential equation $x^2y'' = 2y$ on any interval not containing the point $x = 0$, in particular, on $(0, \infty)$.

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4. Since $y = \sin x + x^2$, we have $y' = \cos x + 2x$ and $y'' = -\sin x + 2$. These functions are defined on $(-\infty, \infty)$. Substituting these expressions into $y'' + y = x^2 + 2$ gives

$$y'' + y = -\sin x + 2 + \sin x + x^2 = x^2 + 2 \qquad \text{for all } x \text{ in } (-\infty, \infty).$$

Therefore, $y = \sin x + x^2$ is a solution to the differential equation on the interval $(-\infty, \infty)$.

6. Differentiating $\theta(t) = 2e^{3t} - e^{2t}$ twice, we get

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{d}{dt}\left(2e^{3t} - e^{2t}\right) = 6e^{3t} - 2e^{2t}, \\ \frac{d^2\theta}{dt^2} &= \frac{d}{dt}\left(6e^{3t} - 2e^{2t}\right) = 18e^{3t} - 4e^{2t}. \end{aligned}$$

So,

$$\frac{d^2\theta}{dt^2} - \theta\,\frac{d\theta}{dt} + 3\theta = -12e^{6t} + 10e^{5t} - 2e^{4t} + 24e^{3t} - 7e^{2t} \not\equiv -2e^{2t}$$

on any interval. Therefore, $\theta(t)$ is not a solution to the given differential equation.

8. We differentiate $y = e^{2x} - 3e^{-x}$ twice:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(e^{2x} - 3e^{-x}\right) = e^{2x}(2) - 3e^{-x}(-1) = 2e^{2x} + 3e^{-x}; \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2e^{2x} + 3e^{-x}\right) = 2e^{2x}(2) + 3e^{-x}(-1) = 4e^{2x} - 3e^{-x}. \end{aligned}$$

Substituting y, y', and y'' into the differential equation, we get

$$\begin{aligned} \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y &= \left(4e^{2x} - 3e^{-x}\right) - \left(2e^{2x} + 3e^{-x}\right) - 2\left(e^{2x} - 3e^{-x}\right) \\ &= (4 - 2 - 2)e^{2x} + (-3 - 3 - 2(-3))e^{-x} = 0. \end{aligned}$$

Hence, $y = e^{2x} - 3e^{-x}$ is an explicit solution to the given differential equation.

10. Differentiating the equation $x^2 + y^2 = 4$ implicitly, we obtain

$$2x + 2yy' = 0 \qquad \Rightarrow \qquad y' = -\frac{x}{y}\,.$$

Since there can be no function $y = f(x)$ that satisfies the differential equation $y' = x/y$ *and* the differential equation $y' = -x/y$ *on the same interval*, we see that $x^2 + y^2 = 4$ does *not* define an implicit solution to the differential equation.

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12. To find dy/dx, we use implicit differentiation.

$$\frac{d}{dx}\left[x^2 - \sin(x+y)\right] = \frac{d}{dx}(1) = 0 \quad \Rightarrow \quad 2x - \cos(x+y)\frac{d}{dx}(x+y) = 0$$
$$\Rightarrow \quad 2x - \cos(x+y)\left(1 + \frac{dy}{dx}\right) = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2x}{\cos(x+y)} - 1 = 2x\sec(x+y) - 1,$$

and so the given differential equation is satisfied.

14. Assuming that C_1 and C_2 are constants, we differentiate the function $\phi(x)$ twice to get

$$\phi'(x) = C_1\cos x - C_2\sin x, \quad \phi''(x) = -C_1\sin x - C_2\cos x.$$

Therefore,

$$\phi'' + \phi = (-C_1\sin x - C_2\cos x) + (C_1\sin x + C_2\cos x) = 0.$$

Thus, $\phi(x)$ is a solution with any choice of constants C_1 and C_2.

16. Differentiating both sides, we obtain

$$\frac{d}{dx}\left(x^2 + Cy^2\right) = \frac{d}{dx}(1) = 0 \qquad \Rightarrow \qquad 2x + 2Cy\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{x}{Cy}.$$

Since, from the given relation, $Cy^2 = 1 - x^2$, we have

$$-\frac{x}{Cy} = \frac{xy}{-Cy^2} = \frac{xy}{x^2-1}.$$

So,

$$\frac{dy}{dx} = \frac{xy}{x^2-1}.$$

Writing $Cy^2 = 1 - x^2$ in the form

$$x^2 + \frac{y^2}{\left(1/\sqrt{C}\right)^2} = 1,$$

we see that the curves defined by the given relation are ellipses with semi-axes 1 and $1/\sqrt{C}$ and so the integral curves are half-ellipses located in the upper/lower half plane.

18. The function $\phi(x)$ is defined and differentiable for all values of x except those satisfying

$$c^2 - x^2 = 0 \qquad \Rightarrow \qquad x = \pm c.$$

In particular, this function is differentiable on $(-c, c)$.

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Clearly, $\phi(x)$ satisfies the initial condition:

$$\phi(0) = \frac{1}{c^2 - 0^2} = \frac{1}{c^2}\,.$$

Next, for any x in $(-c, c)$,

$$\frac{d\phi}{dx} = \frac{d}{dx}\left[\left(c^2 - x^2\right)^{-1}\right] = (-1)\left(c^2 - x^2\right)^{-2}\left(c^2 - x^2\right)' = 2x\left[\left(c^2 - x^2\right)^{-1}\right]^2 = 2x\phi(x)^2.$$

Therefore, $\phi(x)$ is a solution to the equation $y' = 2xy^2$ on $(-c, c)$.

Several integral curves are shown in Fig. **1–A** on page 30.

20. **(a)** Substituting $\phi(x) = e^{mx}$ into the given equation yields

$$\left(e^{mx}\right)'' + 6\left(e^{mx}\right)' + 5\left(e^{mx}\right) = 0 \qquad \Rightarrow \qquad e^{mx}\left(m^2 + 6m + 5\right) = 0.$$

Since $e^{mx} \neq 0$ for any x, $\phi(x)$ satisfies the given equation if and only if

$$m^2 + 6m + 5 = 0 \quad \Leftrightarrow \quad m = -1, -5.$$

(b) We have

$$\left(e^{mx}\right)''' + 3\left(e^{mx}\right)'' + 2\left(e^{mx}\right)' = 0 \qquad \Rightarrow \qquad e^{mx}\left(m^3 + 3m^2 + 2m\right) = 0$$

$$\Rightarrow \qquad m(m^2 + 3m + 2) = 0 \quad \Leftrightarrow \quad m = 0, -1, -2.$$

22. We find

$$\phi'(x) = c_1 e^x - 2c_2 e^{-2x}\,, \qquad \phi''(x) = c_1 e^x + 4c_2 e^{-2x}\,.$$

Substitution yields

$$\begin{aligned} \phi'' + \phi' - 2\phi &= \left(c_1 e^x + 4c_2 e^{-2x}\right) + \left(c_1 e^x - 2c_2 e^{-2x}\right) - 2\left(c_1 e^x + c_2 e^{-2x}\right) \\ &= \left(c_1 + c_1 - 2c_1\right) e^x + \left(4c_2 - 2c_2 - 2c_2\right) e^{-2x} = 0. \end{aligned}$$

Thus, with any choice of constants c_1 and c_2, $\phi(x)$ is a solution to the given equation.

(a) Constants c_1 and c_2 must satisfy the system

$$\begin{cases} 2 = \phi(0) = c_1 + c_2 \\ 1 = \phi'(0) = c_1 - 2c_2\,. \end{cases}$$

Subtracting the second equation from the first one yields

$$3c_2 = 1 \qquad \Rightarrow \qquad c_2 = 1/3 \qquad \Rightarrow \qquad c_1 = 2 - c_2 = 5/3.$$

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(b) Similarly to the part (a), we obtain the system

$$\begin{cases} 1 = \phi(1) = c_1 e + c_2 e^{-2} \\ 0 = \phi'(1) = c_1 e - 2c_2 e^{-2} \end{cases}$$

which has the solution $c_1 = (2/3)e^{-1}$, $c_2 = (1/3)e^2$.

24. In this problem, the independent variable is t, the dependent variable is y. Writing the equation in the form

$$\frac{dy}{dt} = ty + \sin^2 t\,,$$

we conclude that $f(t, y) = ty + \sin^2 t$, $\partial f(t, y)/\partial y = t$. Both functions, f and $\partial f/\partial y$, are continuous on the whole ty-plane. So, Theorem 1 applies for any initial condition, in particular, for $y(\pi) = 5$.

26. With the independent variable t and the dependent variable x, we have

$$f(t, x) = \sin t - \cos x, \qquad \frac{\partial f(t, x)}{\partial x} = \sin x\,,$$

which are continuous on tx-plane. So, Theorem 1 applies for any initial condition.

28. Here, $f(x, y) = 3x - \sqrt[3]{y-1}$ and

$$\frac{\partial f(x, y)}{\partial y} = \frac{\partial}{\partial y}\left[3x - (y-1)^{1/3}\right] = -\frac{1}{3\sqrt[3]{(y-1)^2}}\,.$$

The function f is continuous at any point (x, y) while $\partial f/\partial y$ is defined and continuous at any point (x, y) with $y \neq 1$ i.e., on the xy-plane excluding the horizontal line $y = 1$. Since the initial point $(2, 1)$ belongs to this line, there is no rectangle containing the initial point, on which $\partial f/\partial y$ is continuous. Thus, Theorem 1 does not apply.

30. Here, the initial point (x_0, y_0) is $(0, -1)$ and $G(x, y) = x + y + e^{xy}$. The first partial derivatives,

$$G_x(x, y) = (x + y + e^{xy})'_x = 1 + ye^{xy} \quad \text{and} \quad G_y(x, y) = (x + y + e^{xy})'_y = 1 + xe^{xy},$$

are continuous on the xy-plane. Next,

$$G(0, -1) = -1 + e^0 = 0, \qquad G_y(0, -1) = 1 + (0)e^0 = 1 \neq 0.$$

Therefore, all the hypotheses of Implicit Function Theorem are satisfied, and so the relation $x + y + e^{xy} = 0$ defines a differentiable function $y = \phi(x)$ on some interval $(-\delta, \delta)$ about $x_0 = 0$.

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EXERCISES 1.3: Direction Fields

2. **(a)** For $y = \pm 2x$,

$$\frac{dy}{dx} = \frac{d}{dx}(\pm 2x) = \pm 2 \qquad \text{and} \qquad \frac{4x}{y} = \frac{4x}{\pm 2x} = \pm 2, \quad x \neq 0.$$

Thus $y = 2x$ and $y = -2x$ are solutions to the differential equation $dy/dx = 4x/y$ on any interval not containing the point $x = 0$.

(d) As $x \to \infty$ or $x \to -\infty$, the solution in part (b) increases unboundedly and has the lines $y = 2x$ and $y = -2x$, respectively, as slant asymptotes. The solution in part (c) also increases without bound as $x \to \infty$ and approaches the line $y = 2x$, while it is not even defined for $x < 0$.

4. The direction field and the solution curve satisfying the given initial conditions are sketched in Fig. **1–D** on page 31. From this figure we find that the terminal velocity is $\lim_{t\to\infty} v(t) = 2$.

6. **(a)** The slope of the solution curve to the differential equation $y' = x + \sin y$ at a point (x, y) is given by y'. Therefore the slope at $(1, \pi/2)$ is equal to

$$\left.\frac{dy}{dx}\right|_{x=1} = (x + \sin y)|_{x=1} = 1 + \sin\frac{\pi}{2} = 2.$$

(b) The solution curve is increasing if the slope of the curve is greater than zero. From the part (a), we know that the slope is $x + \sin y$. The function $\sin y$ has values ranging from -1 to 1; therefore if x is greater than 1 then the slope will always have a value greater than zero. This tells us that the solution curve is increasing.

(c) The second derivative of every solution can be determined by differentiating both sides of the original equation, $y' = x + \sin y$. Thus

$$\begin{aligned}
\frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}(x + \sin y) \qquad \Rightarrow \\
\frac{d^2y}{dx^2} &= 1 + (\cos y)\frac{dy}{dx} \qquad \text{(chain rule)} \\
&= 1 + (\cos y)(x + \sin y) \\
&= 1 + x\cos y + \sin y \cos y = 1 + x\cos y + \frac{1}{2}\sin 2y\,.
\end{aligned}$$

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(d) Relative minima occur when the first derivative, y', is equal to zero and the second derivative, y'', is positive (Second Derivative Test). The value of the first derivative at the point $(0,0)$ is given by

$$\frac{dy}{dx} = 0 + \sin 0 = 0.$$

This tells us that the solution has a critical point at the point $(0,0)$. Using the second derivative found in part (c) we have

$$\frac{d^2y}{dx^2} = 1 + 0 \cdot \cos 0 + \frac{1}{2}\sin 0 = 1.$$

This tells us that the point $(0,0)$ is a point of relative minimum.

8. **(a)** For this particle, we have $x(2) = 1$, and so the velocity

$$v(2) = \left.\frac{dx}{dt}\right|_{t=2} = \left. t^3 - x^3 \right|_{t=2} = 2^3 - x(2)^3 = 7.$$

(b) Differentiating the given equation yields

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(t^3 - x^3\right) = 3t^2 - 3x^2\frac{dx}{dt} \\ &= 3t^2 - 3x^2\left(t^3 - x^3\right) = 3t^2 - 3t^3x^2 + 3x^5. \end{aligned}$$

(c) The function u^3 is an increasing function. Therefore, as long as $x(t) < t$, $x(t)^3 < t^3$ and

$$\frac{dx}{dt} = t^3 - x(t)^3 > 0$$

meaning that $x(t)$ increases. At the initial point $t_0 = 2.5$ we have $x(t_0) = 2 < t_0$. Therefore, $x(t)$ cannot take values smaller than 2.5, and the answer is "no".

10. Direction fields and some solution curves to differential equations given in (a)–(e) are shown in Fig. **1–E** through Fig. **1–I** on pages 32–33.

(a) $y' = \sin x$.

(b) $y' = \sin y$.

(c) $y' = \sin x \sin y$.

(d) $y' = x^2 + 2y^2$.

(e) $y' = x^2 - 2y^2$.

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12. The isoclines satisfy the equation $f(x, y) = y = c$, i.e., they are horizontal lines shown in Fig. **1–J**, page 33, along with solution curves. The curve, satisfying the initial condition, is shown in bold.

14. Here, $f(x, y) = x/y$, and so the isoclines are defined by

$$\frac{x}{y} = c \qquad \Rightarrow \qquad y = \frac{1}{c}\,x.$$

These are lines passing through the origin and having slope $1/c$. See Fig. **1–K** on page 34.

16. The relation $x+2y = c$ yields $y = (c-x)/2$. Therefore, the isoclines are lines with slope $-1/2$ and y-intercept $c/2$. See Fig. **1–L** on page 34.

18. The direction field for this equation is shown in Fig. **1–M** on page 34. From this picture we conclude that any solution $y(x)$ approaches zero, as $x \to +\infty$.

EXERCISES 1.4: The Approximation Method of Euler

2. Here $f(x, y) = y(2 - y)$, $x_0 = 0$, and $y_0 = 3$. We again use recursive formulas from Euler's method with $h = 0.1$. Setting $n = 0$, 1, 2, 3, and 4 and rounding off results to three decimal places, we get

$$\begin{aligned}
x_1 &= x_0 + 0.1 = 0.1\,, & y_1 &= y_0 + 0.1 \cdot [y_0(2 - y_0)] = 3 + 0.1 \cdot [3(2 - 3)] = 2.700\,;\\
x_2 &= 0.1 + 0.1 = 0.2\,, & y_2 &= 2.700 + 0.1 \cdot [2.700(2 - 2.700)] = 2.511\,;\\
x_3 &= 0.2 + 0.1 = 0.3\,, & y_3 &= 2.511 + 0.1 \cdot [2.511(2 - 2.511)] \approx 2.383\,;\\
x_4 &= 0.3 + 0.1 = 0.4\,, & y_4 &= 2.383 + 0.1 \cdot [2.383(2 - 2.383)] \approx 2.292\,;\\
x_5 &= 0.4 + 0.1 = 0.5\,, & y_5 &= 2.292 + 0.1 \cdot [2.292(2 - 2.292)] \approx 2.225\,.
\end{aligned}$$

4. In this initial value problem,

$$f(x, y) = \frac{x}{y}\,, \quad x_0 = 0, \quad \text{and} \quad y_0 = -1\,.$$

Thus, with $h = 0.1$, the recursive formulas (2) and (3) become

$$\begin{aligned}
x_{n+1} &= x_n + h = x_n + 0.1\,,\\
y_{n+1} &= y_n + hf(x_n, y_n) = y_n + 0.1\left(\frac{x_n}{y_n}\right), \qquad n = 0, 1, \ldots\,.
\end{aligned}$$

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We set $n = 0$ in these formulas and obtain

$$x_1 = x_0 + 0.1 = 0 + 0.1 = 0.1\,,$$
$$y_1 = y_0 + 0.1 \cdot \left(\frac{x_0}{y_0}\right) = -1 + 0.1\left(\frac{0}{-1}\right) = -1.$$

Putting $n = 1$ in the recursive formulas yields

$$x_2 = x_1 + 0.1 = 0.1 + 0.1 = 0.2\,,$$
$$y_2 = y_1 + 0.1\left(\frac{x_1}{y_1}\right) = -1 + 0.1\left(\frac{0.1}{-1}\right) = -1.01\,.$$

Continuing in the same manner, we find for $n = 2$, 3, and 4:

$$x_3 = 0.2 + 0.1 = 0.3\,, \qquad y_3 = -1.01 + 0.1\left(\frac{0.2}{-1.01}\right) \approx -1.030\,;$$
$$x_4 = 0.3 + 0.1 = 0.4\,, \qquad y_4 = -1.030 + 0.1\left(\frac{0.3}{-1.030}\right) \approx -1.059\,;$$
$$x_5 = 0.4 + 0.1 = 0.5\,, \qquad y_5 = -1.059 + 0.1\left(\frac{0.4}{-1.059}\right) \approx -1.097\,,$$

where we have rounded off all answers to three decimal places.

6. In this problem, $f(x, y) = (y^2 + y)/x$, $x_0 = y_0 = 1$, and $h = 0.2$. The recursive formulas (2) and (3), applied succesively with $n = 1$, 2, 3, and 4, give

$$x_1 = x_0 + 0.2 = 1.2\,, \qquad y_1 = y_0 + 0.2\left(\frac{y_0^2 + y_0}{x_0}\right) = 1 + 0.2\left(\frac{1^2 + 1}{1}\right) = 1.400\,;$$
$$x_2 = 1.2 + 0.2 = 1.4\,, \qquad y_2 = 1.400 + 0.2\left(\frac{1.400^2 + 1.400}{1.2}\right) \approx 1.960\,;$$
$$x_3 = 1.4 + 0.2 = 1.6\,, \qquad y_3 = 1.960 + 0.2\left(\frac{1.960^2 + 1.960}{1.4}\right) \approx 2.789\,;$$
$$x_4 = 1.6 + 0.2 = 1.8\,, \qquad y_4 = 2.789 + 0.2\left(\frac{2.789^2 + 2.789}{1.6}\right) \approx 4.110\,.$$

8. Notice that the independent variable in this problem is t and the dependent variable is x. Hence, the recursive formulas given in equations (2) and (3) become

$$t_{n+1} = t_n + h \qquad \text{and} \qquad \phi(t_{n+1}) \approx x_{n+1} = x_n + hf(t_n, x_n), \qquad n = 0, 1, 2, \ldots\,.$$

We have $f(t, x) = 1 + t\sin(tx)$, $t_0 = 0$, and $x_0 = 0$. Thus, the recursive formula for x has the form

$$x_{n+1} = x_n + h\left[1 + t_n \sin(t_n x_n)\right], \qquad n = 0, 1, 2, \ldots\,.$$

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For the case $N = 1$, we have $h = (1-0)/1 = 1$, which gives us

$$t_1 = 0 + 1 = 1.0\,, \qquad \phi(1) \approx x_1 = 0 + 1.0\,[1 + 0 \cdot \sin 0] = 1\,.$$

For the case $N = 2$, we have $h = 1/2 = 0.5$. Thus, we have

$$t_1 = 0 + 0.5 = 0.5\,, \qquad x_1 = 0 + 0.5\,[1 + 0 \cdot \sin 0] = 0.5\,,$$

and

$$t_2 = 0.5 + 0.5 = 1, \qquad \phi(1) \approx x_2 = 0.5 + 0.5\,[1 + 0.5\sin(0.25)] \approx 1.06185\,.$$

For the case $N = 4$, we have $h = 1/4 = 0.25$, and so the recursive formulas become

$$t_{n+1} = t_n + 0.25 \qquad \text{and} \qquad x_{n+1} = x_n + 0.25\,[1 + t_n \sin(t_n x_n)]\,.$$

Therefore, we have

$$t_1 = 0 + 0.25 = 0.25\,, \qquad x_1 = 0 + 0.25\,[1 + 0 \cdot \sin(0)] = 0.25\,.$$

Plugging these values into the recursive equations yields

$$t_2 = 0.25 + 0.25 = 0.5\,, \qquad x_2 = 0.25 + 0.25\,[1 + 0.25\sin(0.0625)] \approx 0.50390\,.$$

Continuing this way gives us

$$t_3 = 0.75\,, \qquad x_3 = 0.50390 + 0.25\,[1 + 0.5\sin(0.25195)] \approx 0.78507\,,$$
$$t_4 = 1.00\,, \qquad \phi(1) \approx x_4 \approx 1.13920\,.$$

For $N = 8$, we have $h = 1/8 = 0.125$. Thus, the recursive formulas become

$$t_{n+1} = t_n + 0.125 \qquad \text{and} \qquad x_{n+1} = x_n + 0.125\,[1 + t_n \sin(t_n x_n)]\,.$$

Using these formulas and starting with $t_0 = 0$ and $x_0 = 0$, we fill in Table **1–A** on page 29, where the approximations are rounded to five decimal places. From this table we see that $\phi(1) \approx x_8 = 1.19157$.

10. We have $x_0 = 0$ and $y_0 = y\,(x_0) = 0$. Results of computations are shown in Table **1–B** on page 29. We stoppped computations after six steps because the difference between two successive aprroximations was $0.3621 - 0.3561 < 0.01$. Thus, $y(1) \approx 0.3621$.

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Next we check that $y = e^{-x} + x - 1$ is the actual solution to the given initial value problem.

$$y' = \left(e^{-x} + x - 1\right)' = -e^{-x} + 1 = x - \left(e^{-x} + x - 1\right) = x - y,$$
$$y(0) = \left(e^{-x} + x - 1\right)\Big|_{x=0} = e^0 + 0 - 1 = 0.$$

Thus, it is the solution. (The actual value of $y(1)$ is approximately 0.3679 .)

The solution curve $y = e^{-x} + x - 1$ and the polygonal line approximation are shown in Fig. **1–N**, page 34.

Using the Euler's method with $h = 0.1$ we also find that $y(0.6) \approx 0.17 < 0.2$ and $y(0.7) \approx 0.23$. Therefore, within ± 0.05, $x_0 \approx 0.65$.

12. Here, $x_0 = 0$, $y_0 = 1$, $f(x, y) = y$. With $h = 1/n$, the recursive formula (3) of the text yields

$$\begin{aligned} y(1) = y_n = y_{n-1} + \frac{y_{n-1}}{n} &= y_{n-1}\left(1 + \frac{1}{n}\right) = \left[y_{n-2}\left(1 + \frac{1}{n}\right)\right]\left(1 + \frac{1}{n}\right) \\ &= y_{n-2}\left(1 + \frac{1}{n}\right)^2 = \ldots = y_0\left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n . \end{aligned}$$

14. Computation results are given in Table **1–C** on page 30.

16. For this problem notice that the independent variable is t and the dependent variable is T. Hence, in the recursive formulas for Euler's method, t will take the place of x and T will take the place of y. Also we see that $h = 0.1$ and $f(t, T) = K\left(M^4 - T^4\right)$, where $K = 40^{-4}$ and $M = 70$. Therefore, the recursive formulas given in equations (2) and (3) of the text become

$$t_{n+1} = t_n + 0.1\,,$$
$$T_{n+1} = T_n + hf\left(t_n, T_n\right) = T_n + 0.1\left(40^{-4}\right)\left(70^4 - T_n^4\right), \qquad n = 0, 1, 2, \ldots .$$

From the initial condition $T(0) = 100$ we see that $t_0 = 0$ and $T_0 = 100$. Therefore, for $n = 0$, we have

$$t_1 = t_0 + 0.1 = 0 + 0.1 = 0.1\,,$$
$$T_1 = T_0 + 0.1(40^{-4})(70^4 - T_0^4) = 100 + 0.1(40^{-4})(70^4 - 100^4) \approx 97.0316,$$

where we have rounded off to four decimal places.

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For $n = 1$,

$$t_2 = t_1 + 0.1 = 0.1 + 0.1 = 0.2\,,$$

$$T_2 = T_1 + 0.1(40^{-4})(70^4 - T_1^4) = 97.0316 + 0.1(40^{-4})(70^4 - 97.0316^4) \approx 94.5068.$$

By continuing this way, we fill in Table **1–D** on page 30. From this table we see that

$$T(1) = T(t_{10}) \approx T_{10} = 82.694\,,$$
$$T(2) = T(t_{20}) \approx T_{20} = 76.446\,.$$

TABLES

n	t_n	x_n
1	0.125	0.125
2	0.250	0.25024
3	0.375	0.37720
4	0.500	0.50881
5	0.625	0.64954
6	0.750	0.80539
7	0.875	0.98363
8	1.000	1.19157

Table 1–A: Euler's method approximations for the solution of $x' = 1 + t\sin(tx)$, $x(0) = 0$, at $t = 1$ with 8 steps ($h = 1/8$).

h	$y(1)$
1	0
2^{-1}	0.2500
2^{-2}	0.3164
2^{-3}	0.3436
2^{-4}	0.3561
2^{-5}	0.3621

Table 1–B: Euler's approximations to $y' = x - y$, $y(0) = 0$ at $x = 1$ with $h = 2^{-N}$.

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h	$y(2)$
0.5	24.8438
0.1	$\approx 6.4 \cdot 10^{176}$
0.05	$\approx 1.9 \cdot 10^{114571}$
0.01	$> 10^{10^{30}}$

Table 1–C: Euler's method approximations of $y(2)$ for $y' = 2xy^2$, $y(0) = 1$.

n	t_n	T_n	n	t_n	T_n
1	0.1	97.0316	11	1.1	81.8049
2	0.2	94.5068	12	1.2	80.9934
3	0.3	92.3286	13	1.3	80.2504
4	0.4	90.4279	14	1.4	79.5681
5	0.5	88.7538	15	1.5	78.9403
6	0.6	87.2678	16	1.6	78.3613
7	0.7	85.9402	17	1.7	77.8263
8	0.8	84.7472	18	1.8	77.3311
9	0.9	83.6702	19	1.9	76.8721
10	1.0	82.6936	20	2.0	76.4459

Table 1–D: Euler's approximations to the solution of $T' = K\left(M^4 - T^4\right)$, $T(0) = 100$, with $K = 40^{-4}$, $M = 70$, and $h = 0.1$.

FIGURES

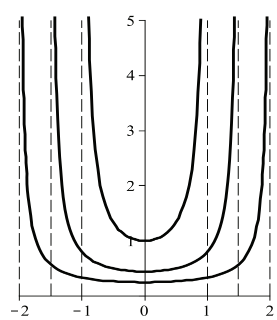

Figure 1–A: Integral curves in Problem 18.

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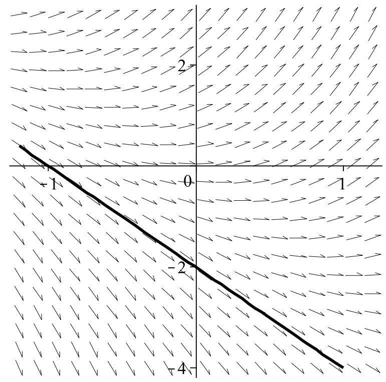

Figure 1–B: The solution curve in Problem 2(a).

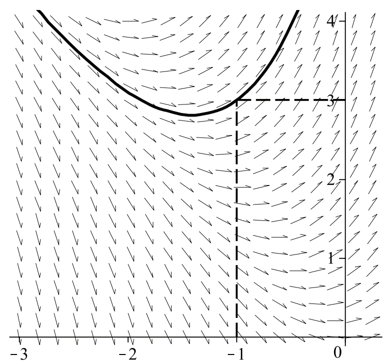

Figure 1–C: The solution curve in Problem 2(b).

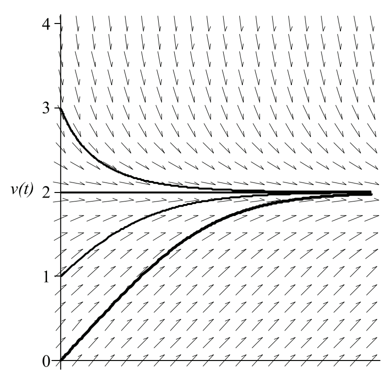

Figure 1–D: The direction field and solution curves in Problem 4.

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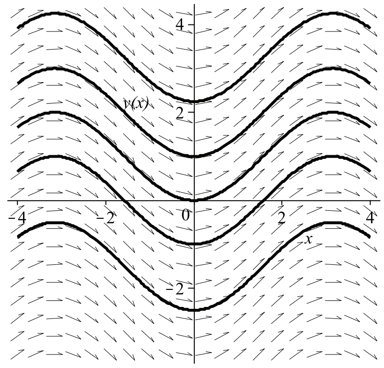

Figure 1–E: The direction field and solution curves in Problem 10(a).

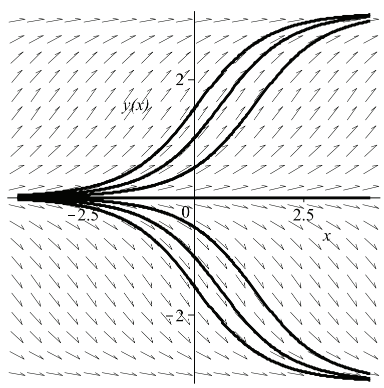

Figure 1–F: The direction field and solution curves in Problem 10(b).

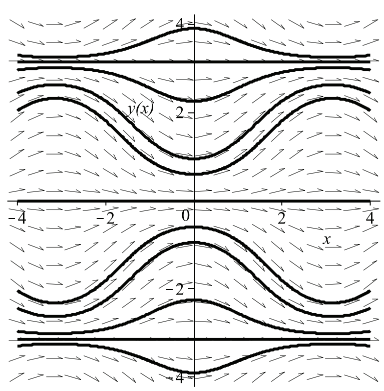

Figure 1–G: The direction field and solution curves in Problem 10(c).

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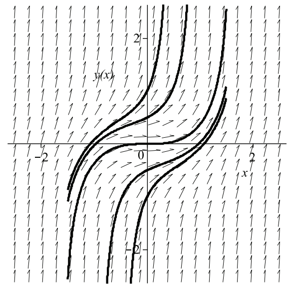

Figure 1–H: The direction field and solution curves in Problem 10(d).

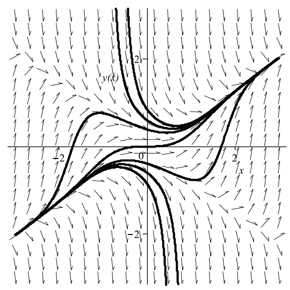

Figure 1–I: The direction field and solution curves in Problem 10(e).

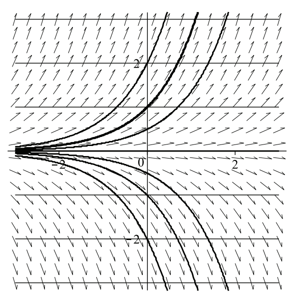

Figure 1–J: The isoclines and solution curves in Problem 12.

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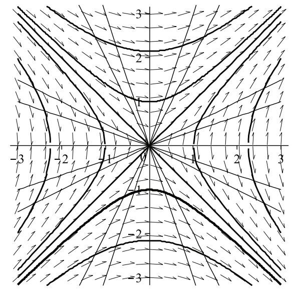

Figure 1–K: The isoclines and solution curves in Problem 14.

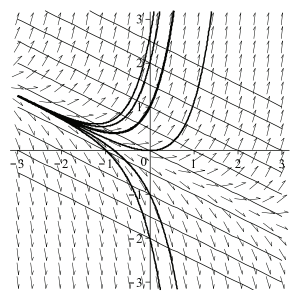

Figure 1–L: The isoclines and solution curves in Problem 16.

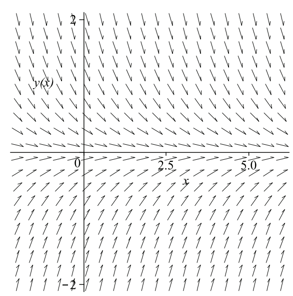

Figure 1–M: The direction field in Problem 18.

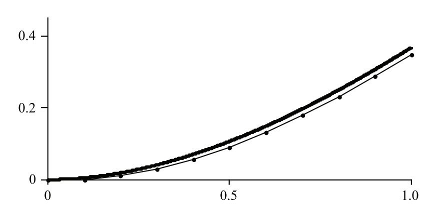

Figure 1–N: Euler's method approximations to $y = e^{-x} + x - 1$ on $[0, 1]$ with $h = 0.1$.

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CHAPTER 2: First Order Differential Equations

EXERCISES 2.2: Separable Equations

2. This equation is separable because we can separate variables by multiplying both sides by dx and dividing by $4y^2 - 3y + 1$.

4. This equation is separable because

$$\frac{dy}{dx} = \frac{ye^{x+y}}{x^2+2} = \left(\frac{e^x}{x^2+2}\right) ye^y = g(x)p(y).$$

6. Writing the equation in the form

$$\frac{ds}{dt} = \frac{s+1}{st} - s^2,$$

we see that the right-hand side cannot be represented in the form $g(t)p(s)$. Therefore, the equation is not separable.

8. Multiplying both sides of the equation by $y^3 dx$ and integrating yields

$$y^3 dy = \frac{dx}{x} \qquad \Rightarrow \qquad \int y^3 dy = \int \frac{dx}{x}$$
$$\Rightarrow \quad \frac{1}{4} y^4 = \ln|x| + C_1 \quad \Rightarrow \quad y^4 = 4\ln|x| + C \quad \Rightarrow \quad y = \pm\sqrt[4]{\ln(x^4) + C}\,,$$

where $C := 4C_1$ is an arbitrary constant.

10. Separating variables and integrating yields

$$xe^{2x} dx = te^{-t} dt \qquad \Rightarrow \qquad \frac{1}{4} e^{2x}(2x-1) = -te^{-t} - e^{-t} + C_1$$
$$\Rightarrow \qquad e^{2x}(2x-1) + 4e^{-t}(t+1) = C$$

where C is an arbitrary constant. (We used integration by parts to evaluate the integrals.)

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12. We have

$$\begin{aligned}
\frac{3v\,dv}{1-4v^2} = \frac{dx}{x} \qquad &\Rightarrow \qquad \int \frac{3v\,dv}{1-4v^2} = \int \frac{dx}{x} \\
\Rightarrow \qquad & -\frac{3}{8}\int \frac{du}{u} = \int \frac{dx}{x} \qquad \left(u = 1-4v^2,\ du = -8v\,dv\right) \\
\Rightarrow \qquad & -\frac{3}{8}\ln\left|1-4v^2\right| = \ln|x| + C_1 \\
\Rightarrow \qquad & 1-4v^2 = \pm\exp\left[-\frac{8}{3}\ln|x| + C_1\right] = Cx^{-8/3},
\end{aligned}$$

where $C = \pm e^{C_1}$ is any nonzero constant. Separating variables, we lost constant solutions satisfying

$$1-4v^2 = 0 \qquad \Rightarrow \qquad v = \pm\frac{1}{2}\,,$$

which can be included in the above formula by letting $C = 0$. Thus,

$$v = \pm\frac{\sqrt{1-Cx^{-8/3}}}{2}\,, \qquad C \text{ arbitrary},$$

is a general solution to the given equation.

14. Separating variables, we get

$$\begin{aligned}
\frac{dy}{\left(1+y^2\right)^{3/2}} = 3x^2dx \qquad &\Rightarrow \qquad \int \frac{dy}{\left(1+y^2\right)^{3/2}} = \int 3x^2dx \\
\Rightarrow \qquad \frac{y}{\sqrt{1+y^2}} = x^3 + C \qquad &\Rightarrow \qquad \frac{y}{\sqrt{1+y^2}} - x^3 = C,
\end{aligned}$$

where C is any constant. To evaluate the first integral, we used the trigonometric substitution $y = \tan t$.

16. We rewrite the equation in the form

$$x(1+y^2)dx + e^{x^2}y\,dy = 0,$$

separate variables, and integrate.

$$\begin{aligned}
e^{-x^2}x\,dx = -\frac{y\,dy}{1+y^2} \qquad &\Rightarrow \qquad \int e^{-x^2}x\,dx = -\int \frac{y\,dy}{1+y^2} \\
\Rightarrow \qquad & \int e^{-u}du = -\frac{dv}{v} \qquad \left(u = x^2,\ v = 1+y^2\right) \\
\Rightarrow \qquad & -e^{-u} = -\ln|v| + C \qquad \Rightarrow \qquad \ln\left(1+y^2\right) - e^{-x^2} = C
\end{aligned}$$

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is an implicit solution to the given equation. Solving for y yields

$$y = \pm\sqrt{C_1 \exp\left[\exp\left(-x^2\right)\right] - 1},$$

where $C_1 = e^C$ is any positive constant,

18. Separating variables yields

$$\frac{dy}{1+y^2} = \tan x dx \quad \Rightarrow \quad \int \frac{dy}{1+y^2} = \int \tan x dx \quad \Rightarrow \quad \arctan y = -\ln|\cos x| + C.$$

Since $y(0) = \sqrt{3}$, we have

$$\arctan\sqrt{3} = -\ln\cos 0 + C = C \qquad \Rightarrow \qquad C = \frac{\pi}{3}.$$

Therefore,

$$\arctan y = -\ln|\cos x| + \frac{\pi}{3} \qquad \Rightarrow \qquad y = \tan\left(-\ln|\cos x| + \frac{\pi}{3}\right)$$

is the solution to the given initial value problem.

20. Separating variables yields

$$(y+1)dy = \frac{4x^2 - x - 2}{x^2(x+1)}\, dx\,. \tag{2.1}$$

For integrating the right-hand side, we use partial fractions.

$$\frac{4x^2 - x - 2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

Solving for A, B, and C, we get $A = 1$, $B = -2$, and $C = 3$. Thus, integrating (2.1), we obtain

$$\frac{1}{2}(y+1)^2 = \ln x + \frac{2}{x} + 3\ln(x+1) + C.$$

With $y(1) = 1$, $C = -3\ln 2$ so that the answer is

$$\frac{1}{2}(y+1)^2 = \ln x + \frac{2}{x} + 3\ln(x+1) - 3\ln 2$$

or, solving for y,

$$y = -1 + \sqrt{\ln\left(x^2\right) + \frac{4}{x} + \ln\left[(x+1)^3\right] - 6\ln 2}\,.$$

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22. Writing $2ydy = -x^2dx$ and integrating, we find

$$y^2 = -\frac{x^3}{3} + C.$$

With $y(0) = 2$,

$$(2)^2 = -\frac{(0)^3}{3} + C \qquad \Rightarrow \qquad C = 4,$$

and so

$$y^2 = -\frac{x^3}{3} + 4 \qquad \Rightarrow \qquad y = \sqrt{-\frac{x^3}{3} + 4}.$$

We note that, taking the square root, we chose the positive sign because $y(0) > 0$.

24. For a general solution, we separate variables and integrate.

$$\int e^{2y} dy = \int 8x^3 dx \qquad \Rightarrow \qquad \frac{e^{2y}}{2} = 2x^4 + C_1 \qquad \Rightarrow \qquad e^{2y} = 4x^4 + C.$$

We substitute now the initial condition, $y(1) = 0$, and obtain

$$1 = 4 + C \qquad \Rightarrow \qquad C = -3.$$

Hence, the answer is given by

$$e^{2y} = 4x^4 - 3 \qquad \Rightarrow \qquad y = \frac{1}{2} \ln\left(4x^4 - 3\right).$$

26. We separate variables and obtain

$$\int \frac{dy}{\sqrt{y}} = -\int \frac{dx}{1+x} \qquad \Rightarrow \qquad 2\sqrt{y} = -\ln|1+x| + C = -\ln(1+x) + C,$$

because at initial point, $x = 0$, $1 + x > 0$. Using the fact that $y(0) = 1$, we find C.

$$2 = 0 + C \qquad \Rightarrow \qquad C = 2,$$

and so $y = [2 - \ln(1+x)]^2/4$ is the answer.

28. We have

$$\frac{dy}{dt} = 2y(1-t) \qquad \Rightarrow \qquad \frac{dy}{y} = 2(1-t)dt \qquad \Rightarrow \qquad \ln|y| = -(t-1)^2 + C$$
$$\Rightarrow \qquad y = \pm e^C e^{-(t-1)^2} = C_1 e^{-(t-1)^2},$$

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where $C_1 \neq 0$ is any constant. Separating variables, we lost the solution $y \equiv 0$, which can be included into the above formula by letting $C_1 = 0$. So, a general solution to the given equation is

$$y = C_1 e^{-(t-1)^2}, \quad C_1 \text{ is an arbitrary constant.}$$

Substituting $t = 0$ and $y = 3$, we find

$$3 = C_1 e^{-1} \qquad \Rightarrow \qquad C_1 = 3e \qquad \Rightarrow \qquad y = 3e^{1-(t-1)^2} = 3e^{2t-t^2}.$$

The graph of this function is given in Fig. **2–A** on page 76.

Since $y(t) > 0$ for any t, from the given equation we have $y'(t) > 0$ for $t < 1$ and $y'(t) < 0$ for $t > 1$. Thus $t = 1$ is the point of absolute maximum with $y_{\max} = y(1) = 3e$.

30. (a) Dividing by $(y+1)^{2/3}$, multiplying by dx, and integrating, we obtain

$$\int \frac{dy}{(y+1)^{2/3}} = \int (x-3)dx \qquad \Rightarrow \qquad 3(y+1)^{1/3} = \frac{x^2}{2} - 3x + C$$

$$\Rightarrow \qquad y = -1 + \left(\frac{x^2}{6} - x + C_1\right)^3.$$

(b) Substituting $y \equiv -1$ into the original equation yields

$$\frac{d(-1)}{dx} = (x-3)(-1+1)^{2/3} = 0,$$

and so the equation is satisfied.

(c) For $y \equiv -1$ for the solution in part (a), we must have

$$\left(\frac{x^2}{6} - x + C_1\right)^3 \equiv 0 \quad \Leftrightarrow \quad \frac{x^2}{6} - x + C_1 \equiv 0,$$

which is impossible since a quadratic polynomial has at most two zeros.

32. (a) The direction field of the given differential equation is shown in Fig. **2–B**, page 76. Using this picture we predict that $\lim_{x\to\infty} \phi(x) = 1$.

(b) In notation of Section 1.4, we have $x_0 = 0$, $y_0 = 1.5$, $f(x, y) = y^2 - 3y + 2$, and $h = 0.1$. With this step size, we need $(1-0)/0.1 = 10$ steps to approximate $\phi(1)$. The results of computation are given in Table **2–A** on page 75. From this table we conclude that $\phi(1) \approx 1.26660$.

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(c) Separating variables and integrating, we obtain

$$\frac{dy}{y^2-3y+2}=dx \quad \Rightarrow \quad \int \frac{dy}{y^2-3y+2}=\int dx \quad \Rightarrow \quad \ln\left|\frac{y-2}{y-1}\right|=x+C\,,$$

where we have used a partial fractions decomposition

$$\frac{1}{y^2-3y+2}=\frac{1}{y-2}-\frac{1}{y-1}$$

to evaluate the integral. The initial condition, $y(0)=1.5$, implies that $C=0$, and so

$$\ln\left|\frac{y-2}{y-1}\right|=x \quad \Rightarrow \quad \left|\frac{y-2}{y-1}\right|=e^x \quad \Rightarrow \quad \frac{y-2}{y-1}=-e^x\,.$$

(We have chosen the negative sign because of the initial condition.) Solving for y yields

$$y=\phi(x)=\frac{e^x+2}{e^x+1}$$

The graph of this solution is shown in Fig. **2–B** on page 76.

(d) We find

$$\phi(1)=\frac{e+2}{e+1}\approx 1.26894\,.$$

Thus, the approximate value $\phi(1)\approx 1.26660$ found in part (b) differs from the actual value by less than 0.003.

(e) We find the limit of $\phi(x)$ at infinity writing

$$\lim_{x\to\infty}\frac{e^x+2}{e^x+1}=\lim_{x\to\infty}\left(1+\frac{1}{e^x+1}\right)=1\,,$$

which confirms our guess in part (a).

34. **(a)** Separating variables and integrating, we get

$$\frac{dT}{T-M}=-kdt \quad \Rightarrow \quad \int \frac{dT}{T-M}=-\int kdt \quad \Rightarrow \quad \ln|T-M|=-kt+C_1$$
$$\Rightarrow \quad |T-M|=e^{C_1}e^{-kt} \quad \Rightarrow \quad T-M=\pm e^{C_1}e^{-kt}=Ce^{-kt},$$

where C is any nonzero constant. We can include the lost solution $T\equiv M$ into this formula by letting $C=0$. Thus, a general solution to the equation is

$$T=M+Ce^{-kt}.$$

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(b) Given that $M = 70°$, $T(0) = 100°$, $T(6) = 80°$, we form a system to determine C and k.

$$\begin{cases} 100 = 70 + C \\ 80 = 70 + Ce^{-6k} \end{cases} \quad \Rightarrow \quad \begin{cases} C = 30 \\ k = -(1/6)\ln[(80-70)/30] = (1/6)\ln 3. \end{cases}$$

Therefore,

$$T = 70 + 30e^{-(t\ln 3)/6} = 70 + (30)3^{-t/6},$$

and after 20 min the reading is

$$T(20) = 70 + (30)3^{-20/6} \approx 70.77°.$$

36. A general solution to the cooling equation found in Problem 34, that is, $T = M + Ce^{-kt}$. Since $T(0) = 100°$, $T(5) = 80°$, and $T(10) = 65°$, we determine M, C, and k from the system

$$\begin{cases} M + C = 100 \\ M + Ce^{-5k} = 80 \\ M + Ce^{-10k} = 65 \end{cases} \quad \Rightarrow \quad \begin{cases} C(1 - e^{-5k}) = 20 \\ Ce^{-5k}(1 - e^{-5k}) = 15 \end{cases} \quad \Rightarrow \quad e^{-5k} = 3/4.$$

To find M, we can now use the first two equations in the above system.

$$\begin{cases} M + C = 100 \\ M + (3/4)C = 80 \end{cases} \quad \Rightarrow \quad M = 20.$$

38. With $m = 100$, $g = 9.8$, and $k = 5$, the equation becomes

$$100\,\frac{dv}{dt} = 100(9.8) - 5v \quad \Rightarrow \quad 20\,\frac{dv}{dt} = 196 - v.$$

Separating variables and integrating yields

$$\int \frac{dv}{v - 196} = -\frac{1}{20}\int dt \quad \Rightarrow \quad \ln|v - 196| = -\frac{t}{20} + C_1 \quad \Rightarrow \quad v = 196 + Ce^{-t/20},$$

where C is an arbitrary nonzero constant. With $C = 0$, this formula also gives the (lost) constant solution $v(t) \equiv 196$. From the initial condition, $v(0) = 10$, we find C.

$$196 + C = 10 \quad \Rightarrow \quad C = -186 \quad \Rightarrow \quad v(t) = 196 - 186e^{-t/20}.$$

The terminal velocity of the object can be found by letting $t \to \infty$.

$$v_\infty = \lim_{t\to\infty}\left(196 - 186e^{-t/20}\right) = 196\,(\text{m/sec}).$$

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40. **(a)** Substituting $\rho = Mp/RT$ into $dp/dz = -\rho g$ yields

$$\frac{dp}{dz} = -\left(\frac{Mp}{RT}\right) g = -\frac{Mg}{RT}\, p.$$

Separating variables and integrating, we find

$$\begin{aligned} \frac{dp}{p} &= -\frac{Mg}{RT}\, dz \qquad (2.2) \\ \Rightarrow \quad \ln p &= -\frac{Mg}{RT}\, z + C_1 \qquad (2.3) \quad \Rightarrow \quad p(z) = Ce^{-(Mg/RT)z}. \end{aligned}$$

For $z = z_0$,

$$p(z_0) = Ce^{-(Mg/RT)z_0} \quad \Rightarrow \quad C = p(z_0)\, e^{(Mg/RT)z_0}.$$

Thus,

$$p(z) = p(z_0)\, e^{(Mg/RT)(z_0 - z)} = p(z_0)\, e^{-Mg(z-z_0)/(RT)}.$$

(b) If $T = T(z)$ varies, then integrating (2.2) from z_0 to z we obtain

$$\begin{aligned} \ln \frac{p(z)}{p(z_0)} &= -\frac{Mg}{R} \int_{z_0}^{z} \frac{d\zeta}{T(\zeta)} \\ \Rightarrow \quad p(z) &= p(z_0) \exp\left[-\frac{Mg}{R} \int_{z_0}^{z} \frac{d\zeta}{T(\zeta)}\right]. \end{aligned}$$

(c) With the given formula for $T(z)$, the integral in part (b) gives

$$\int_{z_0}^{z} \frac{d\zeta}{288 - 0.0065\,(\zeta - z_0)} = -\frac{1}{0.0065} \ln \frac{288 - 0.0065\,(z - z_0)}{288}.$$

Therefore,

$$p(z) = p(z_0) \left[\frac{288 - 0.0065\,(z - z_0)}{288}\right]^{Mg/(0.0065R)}.$$

Substituting the given data into this equation and computing, we get the height

$$z - z_0 \approx 168 \text{ (m)}.$$

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EXERCISES 2.3: Linear Equations

2. Writing

$$\frac{dy}{dx} - x^{-2}y = -x^{-2}\sin x\,,$$

we see that this equation has the form (4) with $P(x) = -x^{-2}$ and $Q(x) = -x^{-2}\sin x$. Hence, it is linear.

Isolating dy/dx yields

$$\frac{dy}{dx} = \frac{y - \sin x}{x^2}.$$

Since the right-hand side cannot be represented as a product $g(x)p(y)$, the equation is not separable.

4. This is a linear equation with independent variable t and dependent variable y since it can be written as

$$\frac{dy}{dt} - \frac{t-1}{t^2+1}\,y = 0\,.$$

This equation is also separable

$$\frac{dy}{dt} = \frac{t-1}{t^2+1}\,y = g(t)p(y).$$

6. In this equation, the independent variable is t and the dependent variable is x. Dividing by x, we obtain

$$\frac{dx}{dt} = \frac{\sin t}{x} - t^2.$$

Therefore, it is neither linear (because of the $(\sin t)/x$ term) nor separable (because the right-hand side is not a product of functions of single variables x and t).

8. This equation can be written as

$$\frac{dy}{dx} - y = e^{3x}\,.$$

Thus, $P(x) \equiv -1$, $Q(x) = e^{3x}$, and the integrating factor is

$$\mu(x) = \exp\left(\int P(x)dx\right) = \exp\left(\int (-1)dx\right) = e^{-x},$$

where we have taken zero integration constant. Multiplying both sides of the given equation by $\mu(x)$, we obtain

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}e^{3x} = e^{2x} \qquad \Rightarrow \qquad \frac{d\left(e^{-x}y\right)}{dx} = e^{2x}\,.$$

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16. Here, $|x| < 1$ and

$$P(x) = -\frac{x^2}{1-x^2} = 1 - \frac{1}{1-x^2}\,, \quad Q(x) = \frac{1+x}{\sqrt{1-x^2}}\,.$$

Therefore,

$$\mu(x) = \exp\left[\int\left(1 - \frac{1}{1-x^2}\right)dx\right] = e^x\sqrt{\frac{1-x}{1+x}}$$

so that $\mu(x)Q(x) = e^x$ and

$$y = e^{-x}\sqrt{\frac{1+x}{1-x}}\,(e^x + C) = \sqrt{\frac{1+x}{1-x}}\left(1 + Ce^{-x}\right).$$

18. Since $\mu(x) = \exp\left(\int 4dx\right) = e^{4x}$, we have

$$\frac{d}{dx}\left(e^{4x}y\right) = e^{4x}e^{-x} = e^{3x}$$

$$\Rightarrow \qquad y = e^{-4x}\int e^{3x}dx = \frac{e^{-x}}{3} + Ce^{-4x}.$$

Substituting the initial condition, $y = 4/3$ at $x = 0$, yields

$$\frac{4}{3} = \frac{1}{3} + C \qquad \Rightarrow \qquad C = 1,$$

and so $y = e^{-x}/3 + e^{-4x}$ is the solution to the given initial value problem.

20. We have

$$\mu(x) = \exp\left(\int\frac{3dx}{x}\right) = \exp\left(3\ln x\right) = x^3$$

$$\Rightarrow \qquad x^3y = \int x^3\left(3x-2\right)dx = \frac{3x^5}{5} - \frac{x^4}{2} + C$$

$$\Rightarrow \qquad y = \frac{3x^2}{5} - \frac{x}{2} + Cx^{-3}.$$

With $y(1) = 1$,

$$1 = y(1) = \frac{3}{5} - \frac{1}{2} + C \qquad \Rightarrow \qquad C = \frac{9}{10} \qquad \Rightarrow \qquad y = \frac{3x^2}{5} - \frac{x}{2} + \frac{9}{10x^3}\,.$$

22. From the standard form of this equation,

$$\frac{dy}{dx} + y\cot x = x,$$

we find

$$\mu(x) = \exp\left(\int\cot x\,dx\right) = \exp\left(\ln\sin x\right) = \sin x.$$

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(Alternatively, one can notice that the left-hand side of the original equation is the derivative of the product $y \sin x$.) So, using integration by parts, we obtain

$$\begin{aligned} y \sin x &= \int x \sin x \, dx = -x \cos x + \sin x + C \\ \Rightarrow \qquad y &= -x \cot x + 1 + C \csc x. \end{aligned}$$

We find C using the initial condition $y(\pi/2) = 2$:

$$2 = -\frac{\pi}{2} \cot \frac{\pi}{2} + 1 + C \csc\left(\frac{\pi}{2}\right) = 1 + C \qquad \Rightarrow \qquad C = 1,$$

and the solution is given by

$$y = -x \cot x + 1 + \csc x.$$

24. **(a)** The equation (12) of the text becomes

$$\begin{aligned} &\frac{dy}{dt} + 20y = 50e^{-10t} \qquad \Rightarrow \qquad \mu(t) = e^{20t} \\ \Rightarrow \qquad & y = e^{-20t} \int e^{20t} \left(50e^{-10t}\right) dt = e^{-20t} \int 50e^{10t} dt = 5e^{-10t} + Ce^{-20t}. \end{aligned}$$

Since $y(0) = 40$, we have

$$40 = 5 + C \qquad \Rightarrow \qquad C = 35 \qquad \Rightarrow \qquad y = 5e^{-10t} + 35e^{-20t}.$$

The term $5e^{-10t}$ will eventually dominate.

(b) This time, the equation (12) has the form

$$\begin{aligned} &\frac{dy}{dt} + 10y = 50e^{-10t} \qquad \Rightarrow \qquad \mu(t) = e^{10t} \\ \Rightarrow \qquad & y = e^{-10t} \int e^{10t} \left(50e^{-10t}\right) dt = e^{-10t}(50t + C). \end{aligned}$$

Substituting the initial condition yields

$$40 = y(0) = C \qquad \Rightarrow \qquad y = e^{-10t}(50t + 40).$$

26. Here

$$\begin{aligned} & P(x) = \frac{\sin x \cos x}{1 + \sin^2 x} \\ \Rightarrow \qquad & \mu(x) = \exp\left(\int \frac{\sin x \cos x \, dx}{1 + \sin^2 x}\right) = \exp\left[\frac{1}{2} \ln\left(1 + \sin^2 x\right)\right] = \sqrt{1 + \sin^2 x}. \end{aligned}$$

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Thus,

$$y\sqrt{1+\sin^2 x} = \int_0^x \sqrt{1+\sin^2 t}\,dt \quad \Rightarrow \quad y = (1+\sin^2 x)^{-1/2}\int_0^x (1+\sin^2 t)^{1/2}dt$$

and

$$y(1) = (1+\sin^2 1)^{-1/2}\int_0^1 (1+\sin^2 t)^{1/2}dt.$$

We now use the Simpson's Rule to find that $y(1) \approx 0.860$.

28. **(a)** Substituting $y = e^{-x}$ into the equation (16) yields

$$\frac{d(e^{-x})}{dx} + e^{-x} = -e^{-x} + e^{-x} = 0.$$

So, $y = e^{-x}$ is a solution to (16).

The function $y = x^{-1}$ is a solution to (17) because

$$\frac{d(x^{-1})}{dx} + (x^{-1})^2 = (-1)x^{-2} + x^{-2} = 0.$$

(b) For any constant C,

$$\frac{d(Ce^{-x})}{dx} + Ce^{-x} = -Ce^{-x} + Ce^{-x} = 0.$$

Thus $y = Ce^{-x}$ is a solution to (16).

Substituting $y = Cx^{-1}$ into (17), we obtain

$$\frac{d(Cx^{-1})}{dx} + (Cx^{-1})^2 = (-C)x^{-2} + C^2x^{-2} = C(C-1)x^{-2},$$

and so we must have $C(C-1) = 0$ in order that $y = Cx^{-1}$ is a solution to (17). Thus, either $C = 0$ or $C = 1$.

(c) For the function $y = C\hat{y}$, one has

$$\frac{d(C\hat{y})}{dx} + P(x)\,(C\hat{y}) = C\frac{d\hat{y}}{dx} + C\,(P(x)\hat{y}) = C\left(\frac{d\hat{y}}{dx} + P(x)\hat{y}\right) = 0$$

if $\hat{y}$ is a solution to $y' + P(x)y = 0$.

30. **(a)** Multiplying both sides of (18) by y^2, we get

$$y^2\frac{dy}{dx} + 2y^3 = x.$$

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If $v = y^3$, then $v' = 3y^2y'$. Thus, $y^2y' = v'/3$, and we have

$$\frac{1}{3}\frac{dv}{dx} + 2v = x,$$

which is equivalent to (19).

(b) The equation (19) is linear with $P(x) = 6$ and $Q(x) = 3x$. So,

$$\mu(x) = \exp\left(\int 6dx\right) = e^{6x}$$

$$\Rightarrow \qquad v(x) = e^{-6x}\int (3xe^{6x})dx = \frac{e^{-6x}}{2}\left(xe^{6x} - \int e^{6x}dx\right)$$

$$= \frac{e^{-6x}}{2}\left(xe^{6x} - \frac{e^{6x}}{6} + C_1\right) = \frac{x}{2} - \frac{1}{12} + Ce^{-6x},$$

where $C = C_1/2$ is an arbitrary constant. The back substitution yields

$$y = \sqrt[3]{\frac{x}{2} - \frac{1}{12} + Ce^{-6x}}.$$

32. In the given equation, $P(x) = 2$, which implies that $\mu(x) = e^{2x}$. Following guidelines, first we solve the equation on $[0, 3]$. On this interval, $Q(x) \equiv 2$. Therefore,

$$y_1(x) = e^{-2x}\int (2)e^{2x}dx = 1 + C_1e^{-2x}.$$

Since $y_1(0) = 0$, we get

$$1 + C_1e^0 = 0 \qquad \Rightarrow \qquad C_1 = -1 \qquad \Rightarrow \qquad y_1(x) = 1 - e^{-2x}.$$

For $x > 3$, $Q(x) = -2$ and so

$$y_2(x) = e^{-2x}\int (-2)e^{2x}dx = -1 + C_2e^{-2x}.$$

We now choose C_2 so that

$$y_2(3) = y_1(3) = 1 - e^{-6} \qquad \Rightarrow \qquad -1 + C_2e^{-6} = 1 - e^{-6} \qquad \Rightarrow \qquad C_2 = 2e^6 - 1.$$

Therefore, $y_2(x) = -1 + (2e^6 - 1)e^{-2x}$, and the continuous solution to the given initial value problem on $[0, \infty)$ is

$$y(x) = \begin{cases} 1 - e^{-2x}, & 0 \le x \le 3, \\ -1 + (2e^6 - 1)e^{-2x}, & x > 3. \end{cases}$$

The graph of this function is shown in Fig. **2–C**, page 76.

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34. **(a)** Since $P(x)$ is continuous on (a,b), its antiderivatives given by $\int P(x)dx$ are continuously differentiable, and therefore continuous, functions on (a,b). Since the function e^x is continuous on $(-\infty,\infty)$, composite functions $\mu(x)=e^{\int P(x)dx}$ are continuous on (a,b). The range of the exponential function is $(0,\infty)$. This implies that $\mu(x)$ is positive with any choice of the integration constant. Using the chain rule, we conclude that

$$\frac{d\mu(x)}{dx}=e^{\int P(x)dx}\frac{d}{dx}\left(\int P(x)dx\right)=\mu(x)P(x)$$

for any x on (a,b).

(b) Differentiating (8), we apply the product rule and obtain

$$\frac{dy}{dx}=-\mu^{-2}\mu'\left(\int \mu Q\,dx+C\right)+\mu^{-1}\mu Q=-\mu^{-1}P\left(\int \mu Q\,dx+C\right)+Q\,,$$

and so

$$\frac{dy}{dx}+Py=\left[-\mu^{-1}P\left(\int \mu Q\,dx+C\right)+Q\right]+P\left[\mu^{-1}\left(\int \mu Q\,dx+C\right)\right]=Q\,.$$

(c) Suggested choice of the antiderivative and the constant C yields

$$y(x_0)=\mu(x)^{-1}\left(\int_{x_0}^{x}\mu Q dx+y_0\mu(x_0)\right)\Bigg|_{x=x_0}=\mu(x_0)^{-1}y_0\mu(x_0)=y_0\,.$$

(d) We assume that $y(x)$ is a solution to the initial value problem (15). Since $\mu(x)$ is a continuous positive function on (a,b), the equation (5) is equivalent to (4). Since, from the part (a), the left-hand side of (5) is the derivative of the product $\mu(x)y(x)$, this function must be an antiderivative of the right-hand side, which is $\mu(x)Q(x)$. Thus, we come up with (8), where the integral means one of the antiderivatives, for example, the one suggested in the part (c) (which has zero value at x_0). Substituting $x=x_0$ into (8), we conclude that

$$y_0=y(x_0)=\mu(x_0)^{-1}\left(\int \mu Q dx+C\right)\Bigg|_{x=x_0}=C\mu(x_0)^{-1},$$

and so $C=y_0\mu(x_0)$ is uniquely defined.

36. **(a)** If $\mu(x)=\exp\left(\int P dx\right)$ and $y_h(x)=\mu(x)^{-1}$, then

$$\frac{dy_h}{dx}=(-1)\mu(x)^{-2}\frac{d\mu(x)}{dx}=-\mu(x)^{-2}\mu(x)P(x)=-\mu(x)^{-1}P(x)$$

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and so

$$\frac{dy_h}{dx} + P(x)y_h = -\mu(x)^{-1}P(x) + P(x)\mu(x)^{-1} = 0,$$

i.e., y_h is a solution to the equation $y' + Py = 0$. Now, the formula (8) yields

$$y = \mu(x)^{-1}\left(\int \mu(x)Q(x)dx + C\right) = y_h(x)v(x) + Cy_h(x) = y_p(x) + Cy_h(x),$$

where $v(x) = \int \mu(x)Q(x)dx$.

(b) Separating variables in (22) and integrating, we obtain

$$\frac{dy}{y} = -\frac{3dx}{x} \qquad \Rightarrow \qquad \int \frac{dy}{y} = -\int \frac{3dx}{x} \qquad \Rightarrow \qquad \ln|y| = -3\ln x + C.$$

Since we need just one solution y_h, we take $C = 0$

$$\ln|y| = -3\ln x \qquad \Rightarrow \qquad y = \pm x^{-3},$$

and we choose, say, $y_h = x^{-3}$.

(c) Substituting $y_p = v(x)y_h(x) = v(x)x^{-3}$ into (21), we get

$$\frac{dv}{dx}y_h + v\frac{dy_h}{dx} + \frac{3}{x}vy_h = \frac{dv}{dx}y_h + v\left(\frac{dy_h}{dx} + \frac{3}{x}y_h\right) = \frac{dv}{dx}y_h = x^2.$$

Therefore, $dv/dx = x^2/y_h = x^5$.

(d) Integrating yields

$$v(x) = \int x^5 dx = \frac{x^6}{6}.$$

(We have chosen zero integration constant.)

(e) The function

$$y = Cy_h + vy_h = Cx^{-3} + \frac{x^3}{6}$$

is a general solution to (21) because

$$\begin{aligned}\frac{dy}{dx} + \frac{3}{x}y &= \frac{d}{dx}\left(Cx^{-3} + \frac{x^3}{6}\right) + \frac{3}{x}\left(Cx^{-3} + \frac{x^3}{6}\right)\\ &= \left(-3Cx^{-4} + \frac{x^2}{2}\right) + \left(3Cx^{-4} + \frac{x^2}{2}\right) = x^2.\end{aligned}$$

38. Dividing both sides of (6) by μ and multiplying by dx yields

$$\begin{aligned}&\frac{d\mu}{\mu} = Pdx \qquad \Rightarrow \qquad \int \frac{d\mu}{\mu} = \int Pdx\\ &\Rightarrow \qquad \ln|\mu| = \int Pdx \qquad \Rightarrow \qquad \mu = \pm\exp\left(\int Pdx\right).\end{aligned}$$

Choosing the positive sign, we obtain (7).

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EXERCISES 2.4: Exact Equations

2. In this equation, $M(x,y)=x^2y+x^4\cos x$ and $N(x,y)=-x^3$. Taking partial derivatives, we obtain

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(x^2y+x^4\cos x\right)=x^2\neq -3x^2=\frac{\partial N}{\partial x}\,.$$

Therefore, according to Theorem 2, the equation is not exact.

Rewriting the equation in the form

$$\frac{dy}{dx}=\frac{x^2y+x^4\cos x}{x^3}=\frac{1}{x}\,y+x\cos x, \tag{2.4}$$

we conclude that it is not separable because the right-hand side in (2.4) cannot be factored as $p(x)q(y)$. We also see that the equation is linear with y as the dependent variable.

4. Here $M(x,y)=ye^{xy}+2x$, $N(x,y)=xe^{xy}-2y$. Thus

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(ye^{xy}+2x\right)=e^{xy}+y\,\frac{\partial}{\partial y}\left(e^{xy}\right)=e^{xy}+ye^{xy}x=e^{xy}(1+yx),\\ \frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(xe^{xy}-2y\right)=e^{xy}+x\,\frac{\partial}{\partial x}\left(e^{xy}\right)=e^{xy}+xe^{xy}y=e^{xy}(1+xy),\end{aligned}$$

Since $\partial M/\partial y=\partial N/\partial x$, the equation is exact.

Writing the equation in the form

$$\frac{dy}{dx}=-\frac{ye^{xy}+2x}{xe^{xy}-2y}\,,$$

we conclude that it is not separable, because the right-hand side cannot be represented as a product of two functions of single variables x and y. Also, the right-hand side is not linear with respect to y, which implies that the equation is not linear with y as the dependent variable. Similarly, choosing x as the dependent variable (taking the reciprocals of both sides), we conclude that the equation is not linear in x either.

6. The differential equation is not separable because $(2xy+\cos y)$ cannot be factored. This equation can be put in standard form by defining x as the dependent variable and y as the independent variable. This gives

$$\frac{dx}{dy}+\frac{2}{y}\,x=\frac{-\cos y}{y^2}\,,$$

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and so we see that it is linear.

If we set $M(x,y) = y^2$ and $N(x,y) = 2xy + \cos y$, we see that the differential equation is also exact, because $M_y(x,y) = 2y = N_x(x,y)$.

8. In this problem, the variables are r and θ, $M(r,\theta) = \theta$, and $N(r,\theta) = 3r - \theta - 1$. Since

$$\frac{\partial M}{\partial \theta} = 1 \neq 3 = \frac{\partial N}{\partial r},$$

the equation is not exact. With r as the dependent variable, the equation takes the form

$$\frac{dr}{d\theta} = -\frac{3r - \theta - 1}{\theta} = -\frac{3}{\theta}r + \frac{\theta + 1}{\theta},$$

and so it is linear. Since the right-hand side in the above equation cannot be factored as $p(\theta)q(r)$, the equation is not separable.

10. We have $M(x,y) = 2xy + 3$ and $N(x,y) = x^2 - 1$. Therefore, $M_y(x,y) = 2x = N_x(x,y)$, and so the equation is exact. We will solve this equation by first integrating $M(x,y)$ with respect to x, although integration of $N(x,y)$ with respect to y is equally easy. Thus,

$$F(x,y) = \int (2xy + 3)\,dx = x^2y + 3x + g(y).$$

Differentiating $F(x,y)$ with respect to y gives

$$F_y(x,y) = x^2 + g'(y) = N(x,y) = x^2 - 1.$$

From this equation we see that $g' = -1$. Integrating yields

$$g(y) = \int (-1)\,dy = -y.$$

Since the constant of integration will be incorporated into the parameter of the solution, it is not written here. Substituting the expression for $g(y)$ into the formula that we found for $F(x,y)$ yields

$$F(x,y) = x^2y + 3x - y.$$

Therefore, a general solution to the given differential equation is

$$x^2y + 3x - y = C \qquad \Rightarrow \qquad y = \frac{C - 3x}{x^2 - 1}.$$

Note: *The given equation could be solved by the method of grouping.* Indeed, writing

$$(2xy\,dx + x^2\,dy) + (3\,dx - dy) = 0,$$

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we recognize the first term of the left-hand side as the total differential of x^2y, and the second term is the total differential of $(3x - y)$. Thus, we again find that

$$F(x, y) = x^2y + 3x - y\,.$$

12. We compute

$$\frac{\partial M}{\partial y} = e^x \cos y = \frac{\partial N}{\partial x}\,.$$

Thus, the equation is exact.

$$F(x, y) = \int \left(e^x \sin y - 3x^2\right) dx = e^x \sin y - x^3 + g(y),$$
$$\frac{\partial F}{\partial y} = e^x \cos y + g'(y) = e^x \cos y - \frac{1}{3}\, y^{-2/3}$$
$$\Rightarrow \qquad g'(y) = -\frac{1}{3}\, y^{-2/3} \qquad \Rightarrow \qquad g(y) = \frac{1}{3} \int y^{-2/3} dy = y^{1/3}\,.$$

So, $e^x \sin y - x^3 + \sqrt[3]{y} = C$ is a general solution.

14. Since $M(t, y) = e^t(y - t)$, $N(t, y) = 1 + e^t$, we find that

$$\frac{\partial M}{\partial y} = e^t = \frac{\partial N}{\partial t}.$$

Then

$$F(t, y) = \int (1 + e^t) dy = (1 + e^t)y + h(t),$$
$$\frac{\partial F}{\partial t} = e^t y + h'(t) = e^t(y - t) \qquad \Rightarrow \qquad h\prime(t) = -te^t$$
$$\Rightarrow \qquad h(t) = -\int te^t dt = -(t - 1)e^t,$$

and a general solution is given by

$$(1 + e^t)y - (t - 1)e^t = C \qquad \Rightarrow \qquad y = \frac{(t - 1)e^t + C}{1 + e^t}\,.$$

16. Computing

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(ye^{xy} - y^{-1}\right) = e^{xy} + xye^{xy} + y^{-2},$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(xe^{xy} + xy^{-2}\right) = e^{xy} + xye^{xy} + y^{-2},$$

we see that the equation is exact. Therefore,

$$F(x, y) = \int \left(ye^{xy} - y^{-1}\right) dx = e^{xy} - xy^{-1} + g(y).$$

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So,

$$\frac{\partial F}{\partial y} = xe^{xy} + xy^{-2} + g'(y) = N(x,y) \qquad \Rightarrow \qquad g'(y) = 0.$$

Thus, $g(y) = 0$, and the answer is $e^{xy} - xy^{-1} = C$.

18. Since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y^2 + \sin(x+y),$$

the equation is exact. We find

$$F(x,y) = \int \left[2x + y^2 - \cos(x+y)\right] dx = x^2 + xy^2 - \sin(x+y) + g(y),$$
$$\frac{\partial F}{\partial y} = 2xy - \cos(x+y) + g'(y) = 2xy - \cos(x+y) - e^y$$
$$\Rightarrow \qquad g'(y) = -e^y \qquad \Rightarrow \qquad g(y) = -e^y.$$

Therefore,

$$F(x,y) = x^2 + xy^2 - \sin(x+y) - e^y = C$$

gives a general solution.

20. We find

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[y\cos(xy)\right] = \cos(xy) - xy\sin(xy),$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[x\cos(xy)\right] = \cos(xy) - xy\sin(xy).$$

Therefore, the equation is exact and

$$F(x,y) = \int \left(x\cos(xy) - y^{-1/3}\right) dy = \sin(xy) - \frac{3}{2}\,y^{2/3} + h(x)$$
$$\frac{\partial F}{\partial x} = y\cos(xy) + h'(x) = \frac{2}{\sqrt{1-x^2}} + y\cos(xy)$$
$$\Rightarrow \qquad h'(x) = \frac{2}{\sqrt{1-x^2}} \qquad \Rightarrow \qquad h(x) = 2\arcsin x,$$

and a general solution is given by

$$\sin(xy) - \frac{3}{2}\,y^{2/3} + 2\arcsin x = C.$$

22. In Problem 16, we found that a general solution to this equation is

$$e^{xy} - xy^{-1} = C.$$

Substituting the initial condition, $y(1) = 1$, yields $e - 1 = C$. So, the answer is

$$e^{xy} - xy^{-1} = e - 1.$$

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24. First, we check the given equation for exactness.

$$\frac{dM}{dx} = e^t = \frac{\partial N}{\partial t}\,.$$

So, it is exact. We find

$$F(t,x) = \int \left(e^t - 1\right) dx = x\left(e^t - 1\right) + g(t),$$
$$\frac{\partial F}{\partial t} = xe^t + g'(t) = xe^t + 1 \qquad \Rightarrow \qquad g(t) = \int dt = t$$
$$\Rightarrow \qquad x\left(e^t - 1\right) + t = C$$

is a general solution. With $x(1) = 1$, we get

$$(1)\,(e - 1) + 1 = C \qquad \Rightarrow \qquad C = e,$$

and the solution is given by

$$x = \frac{e - t}{e^t - 1}\,.$$

26. Taking partial derivatives M_y and N_x, we find that the equation is exact. So,

$$F(x,y) = \int (\tan y - 2)\,dx = x(\tan y - 2) + g(y),$$
$$\frac{\partial F}{\partial y} = x\sec^2 y + g'(y) = x\sec^2 y + y^{-1}$$
$$\Rightarrow \qquad g'(y) = y^{-1} \qquad \Rightarrow \qquad g(y) = \ln|y|,$$

and

$$x(\tan y - 2) + \ln|y| = C$$

is a general solution. Substituting $y(0) = 1$ yields $C = 0$. Therefore, the answer is

$$x(\tan y - 2) + \ln y = 0.$$

(We removed the absolute value sign in the logarithmic function because $y(0) > 0$.)

28. **(a)** Computing

$$\frac{\partial M}{\partial y} = \cos(xy) - xy\sin(xy),$$

which must be equal to $\partial N/\partial x$, we find that

$$\begin{aligned} N(x,y) &= \int [\cos(xy) - xy\sin(xy)]\,dx \\ &= \int [x\cos(xy)]'_x\,dx = x\cos(xy) + g(y). \end{aligned}$$

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(b) Since

$$\frac{\partial M}{\partial y} = (1+xy)e^{xy} - 4x^3 = \frac{\partial N}{\partial x}\,,$$

we conclude that

$$N(x,y) = \int \left[(1+xy)e^{xy} - 4x^3\right] dx = xe^{xy} - x^4 + g(y).$$

30. **(a)** Differentiating, we find that

$$\frac{\partial M}{\partial y} = 5x^2 + 12x^3y + 8xy\,,$$
$$\frac{\partial N}{\partial x} = 6x^2 + 12x^3y + 6xy\,.$$

Since $M_y \neq N_x$, the equation is not exact.

(b) Multiplying given equation by $x^n y^m$ and taking partial derivatives of new coefficients yields

$$\frac{d}{dy}\left(5x^{n+2}y^{m+1} + 6x^{n+3}y^{m+2} + 4x^{n+1}y^{m+2}\right)$$
$$= 5(m+1)x^{n+2}y^m + 6(m+2)x^{n+3}y^{m+1} + 4(m+2)x^{n+1}y^{m+1}$$
$$\frac{d}{dx}\left(2x^{n+3}y^m + 3x^{n+4}y^{m+1} + 3x^{n+2}y^{m+1}\right)$$
$$= 2(n+3)x^{n+2}y^m + 3(n+4)x^{n+3}y^{m+1} + 3(n+2)x^{n+1}y^{m+1}.$$

In order that these polynomials are equal, we must have equal coefficients at similar monomials. Thus, n and m must satisfy the system

$$\begin{cases} 5(m+1) = 2(n+3) \\ 6(m+2) = 3(n+4) \\ 4(m+2) = 3(n+2). \end{cases}$$

Solving, we obtain $n = 2$ and $m = 1$. Therefore, multiplying the given equation by x^2y yields an exact equation.

(c) We find

$$\begin{aligned} F(x,y) &= \int \left(5x^4y^2 + 6x^5y^3 + 4x^3y^3\right) dx \\ &= x^5y^2 + x^6y^3 + x^4y^3 + g(y). \end{aligned}$$

Therefore,

$$\frac{\partial F}{\partial y} = 2x^5y + 3x^6y^2 + 3x^4y^2 + g'(y)$$

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$$= \quad 2x^5y + 3x^6y^2 + 3x^4y^2 \qquad \Rightarrow \qquad g(y) = 0,$$

and a general solution to the given equation is

$$x^5y^2 + x^6y^3 + x^4y^3 = C.$$

32. **(a)** The slope of the orthogonal curves, say $m_\perp$, must be $-1/m$, where m is the slope of the original curves. Therefore, we have

$$m_\perp = \frac{F_y(x,y)}{F_x(x,y)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{F_y(x,y)}{F_x(x,y)} \quad \Rightarrow \quad F_y(x,y)\,dx - F_x(x,y)\,dy = 0.$$

(b) Let $F(x,y) = x^2 + y^2$. Then we have $F_x(x,y) = 2x$ and $F_y(x,y) = 2y$. Substituting these expressions into the result of part (a) gives us

$$2y\,dx - 2x\,dy = 0 \qquad \Rightarrow \qquad y\,dx - x\,dy = 0.$$

To find the orthogonal trajectories, we must solve this differential equation. To this end, note that this equation is separable and thus

$$\int \frac{1}{x}\,dx = \int \frac{1}{y}\,dy \qquad \Rightarrow \qquad \ln|x| = \ln|y| + C$$

$$\Rightarrow \qquad e^{\ln|x|-C} = e^{\ln|y|} \qquad \Rightarrow \qquad y = kx, \quad \text{where } k = \pm e^{-C}.$$

Therefore, the orthogonal trajectories are lines through the origin.

(c) Let $F(x,y) = xy$. Then we have $F_x(x,y) = y$ and $F_y(x,y) = x$. Substituting these expressions into the result of part (a) yields

$$x\,dx - y\,dy = 0.$$

To find the orthogonal trajectories, we must solve this differential equation. To this end, note that this equation is separable and thus

$$\int x\,dx = \int y\,dy \qquad \Rightarrow \qquad \frac{x^2}{2} = \frac{y^2}{2} + C \qquad \Rightarrow \qquad x^2 - y^2 = k\,,$$

where $k := 2C$. Therefore, the orthogonal trajectories are hyperbolas.

34. To use the method described in Problem 32, we rewrite the equation $x^2 + y^2 = kx$ in the form $x + x^{-1}y^2 = k$. Thus, $F(x,y) = x + x^{-1}y^2$,

$$\frac{\partial F}{\partial x} = 1 - x^{-2}y^2, \qquad \frac{\partial F}{\partial y} = 2x^{-1}y.$$

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Substituting these derivatives in the equation given in Problem 32(b), we get the required. Multiplying the equation by $x^n y^m$, we obtain

$$2x^{n-1}y^{m+1}dx + \left(x^{n-2}y^{m+2} - x^n y^m\right) dy = 0.$$

Therefore,

$$\frac{\partial M}{\partial y} = 2(m+1)x^{n-1}y^m\,,$$
$$\frac{\partial N}{\partial x} = (n-2)x^{n-3}y^{m+2} - nx^{n-1}y^m\,.$$

Thus, to have an exact equation, n and m must satisfy

$$\begin{cases} n-2=0 \\ 2(m+1) = -n\,. \end{cases}$$

Solving, we obtain $n = 2$, $m = -2$. With this choice, the equation becomes

$$2xy^{-1}dx + \left(1 - x^2y^{-2}\right) dy = 0,$$

and so

$$G(x,y) = \int M(x,y)dx = \int 2xy^{-1}dx = x^2y^{-1} + g(y),$$
$$\frac{\partial G}{\partial y} = -x^2y^{-2} + g'(y) = N(x,y) = 1 - x^2y^{-2}.$$

Therefore, $g(y) = y$, and the family of orthogonal trajectories is given by $x^2y^{-1}+y = C$. Writing this equation in the form $x^2+y^2-Cy = 0$, we see that, given C, the trajectory is the circle centered at $(0, C/2)$ and of radius $C/2$.

Several given curves and their orthogonal trajectories are shown in Fig. **2–D**, page 76.

36. The first equation in (4) follows from (9) and the Fundamental Theorem of Calculus.

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left[\int_{x_0}^{x} M(t,y)dt + g(y)\right] = M(t,y)|_{t=x} = M(x,y).$$

For the second equation in (4),

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int_{x_0}^{x} M(t,y)dt + g(y)\right]$$

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$$= \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)dt + g'(y)$$

$$= \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)dt + \left[N(x,y) - \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)dt \right] = N(x,y).$$

EXERCISES 2.5: Special Integrating Factors

2. Here, $M(x,y) = 2y^3 + 2y^2$ and $N(x,y) = 3y^2x + 2xy$. Computing

$$\frac{\partial M}{\partial y} = 6y^2 + 4y \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 3y^2 + 2y\,,$$

we conclude that this equation is not exact. Note that these derivatives, as well as M itself, depend on y only. Then, clearly, so does the expression $(\partial N/\partial x - \partial M/\partial y)/M$, and the given equation has an integrating factor depending on y alone. Also, since

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{(6y^2+4y)-(3y^2+2y)}{3y^2x+2xy} = \frac{3y^2+2y}{x(3y^2+2y)} = \frac{1}{x}\,,$$

the equation has an integrating factor depending on x only.

Writing the equation in the form

$$\frac{dx}{dy} = -\frac{3y^2x+2xy}{2y^3+y^2} = -\frac{xy(3y+2)}{2y^2(y+1)} = -\frac{y(3y+2)}{2y^2(y+1)}\,x\,,$$

we conclude that it is separable and linear with x as the dependent variable.

4. This equation is not separable, because of the factor (y^2+2xy). It is not linear in either variable because of the terms y^2 and x^2. To see if it is exact, we compute $M_y(x,y)$ and $N_x(x,y)$, and find that

$$M_y(x,y) = 2y + 2x \neq -2x = N_x(x,y).$$

Therefore, the equation is not exact. To see if we can find an integrating factor of the form $\mu(x)$, we compute

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{2y+4x}{-x^2}\,,$$

which is not a function of x alone. To see if we can find an integrating factor of the form $\mu(y)$, we compute

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{-4x-2y}{y^2+2xy} = \frac{-2(2x+y)}{y(y+2x)} = \frac{-2}{y}\,.$$

Thus the equation has an integrating factor that is a function of y alone.

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6. In this problem, $M(x,y) = 2y^2x - y$ and $N(x,y) = x$. Therefore,

$$\frac{\partial M}{\partial y} = 4yx - 1 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 1 \qquad \Rightarrow \qquad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 4yx\,.$$

The equation is not exact, because $\partial M/\partial y \neq \partial N/\partial x$, but it has an integrating factor, which depends on y because

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{2 - 4yx}{2y^2x - y} = \frac{-2(2yx-1)}{y(2yx-1)} = \frac{-2}{y}\,.$$

Isolating dy/dx, we obtain

$$\frac{dy}{dx} = \frac{y - 2y^2x}{x} = \frac{y}{x} - 2y^2\,.$$

The right-hand side cannot be written as $p(x)q(y)$, so the equation is not separable. Also, it is not linear with y as the dependent variable (because of $2y^2$ term). Taking the reciprocals, we conclude that it is not linear with the dependent variable x.

8. The equation $(3x^2 + y)\,dx + (x^2y - x)\,dy = 0$ is not separable or linear. To see if it is exact, we compute

$$\frac{\partial M}{\partial y} = 1 \neq 2xy - 1 = \frac{\partial N}{\partial x}\,.$$

Thus, the equation is not exact. To see if we can find an integrating factor, we compute

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{-2(xy-1)}{x(xy-1)} = \frac{-2}{x}\,.$$

Thus, an integrating factor $\mu(x)$ is given by

$$\mu(x) = \exp\left(\int \frac{-2}{x}\,dx\right) = \exp\left(-2\ln|x|\right) = x^{-2}.$$

To solve the equation, we multiply it by the integrating factor x^{-2} and get

$$(3 + yx^{-2})\,dx + (y - x^{-1})\,dy = 0.$$

This new equation is now exact. Thus, we find $F(x,y)$ integrating $M(x,y) = 3 + yx^{-2}$ with respect to x.

$$F(x,y) = \int \left(3 + yx^{-2}\right) dx = 3x - yx^{-1} + g(y)$$

$$\Rightarrow \qquad F_y(x,y) = -x^{-1} + g'(y) = N(x,y) = y - x^{-1}$$

$$\Rightarrow \qquad g'(y) = y \qquad \Rightarrow \qquad g(y) = \frac{y^2}{2}\,.$$

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Therefore,

$$F(x,y) = 3x - yx^{-1} + \frac{y^2}{2},$$

and an implicit solution to the given equation is

$$\frac{y^2}{2} - \frac{y}{x} + 3x = C.$$

Since $\mu(x) = x^{-2}$ we must check if the solution $x \equiv 0$ was lost. The function $x \equiv 0$ is a solution to the original equation, but is not given by the above implicit solution for any choice of C. Hence,

$$\frac{y^2}{2} - \frac{y}{x} + 3x = C \qquad \text{and} \qquad x \equiv 0$$

give a general solution.

10. We compute the partial derivatives of $M(x,y) = 2y^2 + 2y + 4x^2$ and $N(x,y) = 2xy + x$.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2y^2 + 2y + 4x^2\right) = 4y + 2, \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2xy + x) = 2y + 1.$$

Although the equation is not exact ($\partial M/\partial y \neq \partial N/\partial x$), the quotient

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{(4y+2) - (2y+1)}{2xy + x} = \frac{2y+1}{x(2y+1)} = \frac{1}{x}$$

depends on x only, and so the equation has an integrating factor, which can be found by applying formula (8). Namely,

$$\mu(x) = \exp\left(\int \frac{1}{x}\,dx\right) = \exp\left(\ln|x|\right) = |x|.$$

Note that if μ is an integrating factor, then $-\mu$ is an integrating factor as well. This observation allows us to take $\mu(x) = x$. Multiplying the given differential equation by x yields an exact equation

$$\left(2y^2 + 2y + 4x^2\right) x\,dx + x^2\,(2y+1)\;dy = 0.$$

Therefore,

$$\begin{aligned}
& F(x,y) = \int x^2(2y+1)\,dy = x^2\left(y^2 + y\right) + h(x) \\
\Rightarrow \quad & \frac{\partial F}{\partial x} = 2x\left(y^2 + y\right) + h'(x) = \left(2y^2 + 2y + 4x^2\right) x \\
\Rightarrow \quad & h'(x) = 4x^3 \qquad \Rightarrow \qquad h(x) = \int 4x^3 dx = x^4 \\
\Rightarrow \quad & F(x,y) = x^2\left(y^2 + y\right) + x^4 = x^2y^2 + x^2y + x^4,
\end{aligned}$$

and $x^2y^2 + x^2y + x^4 = C$ is a general solution.

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12. Here, $M(x,y) = 2xy^3 + 1$, $N(x,y) = 3x^2y^2 - y^{-1}$. Since

$$\frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x},$$

the equation is exact. So, we find that

$$\begin{aligned} F(x,y) &= \int \left(2xy^3 + 1\right) dx = x^2y^3 + x + g(y), \\ \frac{\partial F}{\partial y} &= 3x^2y^2 + g'(y) = 3x^2y^2 - y^{-1} \\ \Rightarrow \qquad g'(y) &= -y^{-1} \qquad \Rightarrow \qquad g(y) = -\ln|y|, \end{aligned}$$

and the given equation has a general solution

$$x^2y^3 + x - \ln|y| = C.$$

14. Multiplying the given equation by x^ny^m yields

$$\left(12x^ny^m + 5x^{n+1}y^{m+1}\right) dx + \left(6x^{n+1}y^{m-1} + 3x^{n+2}y^m\right) dy = 0.$$

Therefore,

$$\begin{aligned} \frac{\partial M}{\partial y} &= 12mx^ny^{m-1} + 5(m+1)x^{n+1}y^m, \\ \frac{\partial N}{\partial x} &= 6(n+1)x^ny^{m-1} + 3(n+2)x^{n+1}y^m. \end{aligned}$$

Matching the coefficients, we get a system

$$\begin{cases} 12m = 6(n+1) \\ 5(m+1) = 3(n+2) \end{cases}$$

to determine n and m. This system has the solution $n = 3$, $m = 2$. Thus, the given equation multiplied by x^3y^2, that is,

$$\left(12x^3y^2 + 5x^4y^3\right) dx + \left(6x^4y + 3x^5y^2\right) dy = 0,$$

is exact. We compute

$$\begin{aligned} F(x,y) &= \int \left(12x^3y^2 + 5x^4y^3\right) dx = 3x^4y^2 + x^5y^3 + g(y), \\ \frac{\partial F}{\partial y} &= 6x^4y + 3x^5y^2 + g'(y) = 6x^4y + 3x^5y^2 \\ \Rightarrow \qquad g'(y) &= 0 \qquad \Rightarrow \qquad g(y) = 0, \end{aligned}$$

and so $3x^4y^2 + x^5y^3 = C$ is a general solution to the given equation.

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16. (a) An equation $Mdx + Ndy = 0$ has an integrating factor $\mu(x + y)$ if and only if the equation

$$\mu(x+y)M(x,y)dx + \mu(x+y)N(x,y)dy = 0$$

is exact. According to Theorem 2, Section 2.4, this means that

$$\frac{\partial}{\partial y}\left[\mu(x+y)M(x,y)\right] = \frac{\partial}{\partial x}\left[\mu(x+y)N(x,y)\right].$$

Applying the product and chain rules yields

$$\mu'(x+y)M(x,y) + \mu(x+y)\frac{\partial M(x,y)}{\partial y} = \mu'(x+y)N(x,y) + \mu(x+y)\frac{\partial N(x,y)}{\partial x}.$$

Collecting similar terms yields

$$\mu'(x+y)\left[M(x,y) - N(x,y)\right] = \mu(x+y)\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$$

$$\Leftrightarrow \qquad \frac{\partial N/\partial x - \partial M/\partial y}{M - N} = \frac{\mu'(x+y)}{\mu(x+y)}. \tag{2.5}$$

The right-hand side of (2.5) depends on $x + y$ only so the left-hand side does.

To find an integrating factor, we let $s = x + y$ and denote

$$G(s) = \frac{\partial N/\partial x - \partial M/\partial y}{M - N}.$$

Then (2.5) implies that

$$\frac{\mu'(s)}{\mu(s)} = G(s) \qquad \Rightarrow \qquad \ln|\mu(s)| = \int G(s)\,ds$$

$$\Rightarrow \quad |\mu(s)| = \exp\left[\int G(s)\,ds\right] \quad \Rightarrow \quad \mu(s) = \pm\exp\left[\int G(s)\,ds\right]. \tag{2.6}$$

In this formula, we can choose either sign and any integration constant.

(b) We compute

$$\frac{\partial N/\partial x - \partial M/\partial y}{M - N} = \frac{(1+y) - (1+x)}{(3+y+xy) - (3+x+xy)} = 1.$$

Applying formula (2.6), we obtain

$$\mu(s) = \exp\left[\int (1)ds\right] = e^s \qquad \Rightarrow \qquad \mu(x+y) = e^{x+y},$$

Multiplying the given equation by $\mu(x + y)$, we get an exact equation

$$e^{x+y}(3+y+xy)dx + e^{x+y}(3+x+xy)dy = 0$$

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and follow the procedure of solving exact equations, Section 2.4.

$$\begin{aligned} F(x,y) &= \int e^{x+y}(3+y+xy)\,dx = e^{y}\left[(3+y)\int e^{x}dx + y\int xe^{x}dx\right] \\ &= e^{y}\left[(3+y)e^{x} + y(x-1)e^{x}\right] + h(y) = e^{x+y}(3+xy) + h(y)\,. \end{aligned}$$

Taking the partial derivative of F with respect to y, we find $h(y)$.

$$\begin{aligned} &\frac{\partial F}{\partial y} = e^{x+y}(3+xy+x) + h'(y) = e^{x+y}N(x,y) = e^{x+y}(3+x+xy) \\ &\Rightarrow \qquad h'(y) = 0 \qquad \Rightarrow \qquad h(y) = 0\,. \end{aligned}$$

Thus, a general solution is

$$e^{x+y}(3+xy) = C\,.$$

18. The given condition, $xM(x,y) + yN(x,y) \equiv 0$, is equivalent to $yN(x,y) \equiv -xM(x,y)$. In particular, substituting $x = 0$, we obtain

$$yN(0,y) \equiv -(0)M(0,y) \equiv 0.$$

This implies that $x \equiv 0$ is a solution to the given equation.

To obtain other solutions, we multiply the equation by $x^{-1}y$. This gives

$$\begin{aligned} x^{-1}yM(x,y)dx + x^{-1}yN(x,y)dy &= x^{-1}yM(x,y)dx - x^{-1}xM(x,y)dy \\ &= xM(x,y)\left(x^{-2}ydx - x^{-1}dy\right) = -xM(x,y)d\left(x^{-1}y\right) = 0\,. \end{aligned}$$

Therefore, $x^{-1}y = C$ or $y = Cx$.

Thus, a general solution is

$$y = Cx \quad \text{and} \quad x \equiv 0.$$

20. For the equation

$$e^{\int P(x)dx}\left[P(x)y - Q(x)\right]dx + e^{\int P(x)dx}dy = 0,$$

we compute

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(e^{\int P(x)dx}\left[P(x)y - Q(x)\right]\right) = e^{\int P(x)dx}P(x), \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(e^{\int P(x)dx}\right) = e^{\int P(x)dx}\frac{d}{dx}\left(\int P(x)dx\right) = e^{\int P(x)dx}P(x). \end{aligned}$$

Therefore, $\partial M/\partial y = \partial N/\partial x$, and the equation is exact.

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EXERCISES 2.6: Substitutions and Transformations

2. We can write the equation in the form

$$\frac{dy}{dx} = (y - 4x - 1)^2 = [(y - 4x) - 1]^2 = G(y - 4x),$$

where $G(t) = (t - 1)^2$. Thus, it is of the form $dy/dx = G(ax + by)$.

4. In this equation, the variables are x and t. Its coefficients, $t + x + 2$ and $3t - x - 6$, are linear functions of x and t. Therefore, the given equation is an equation with linear coefficients.

6. The given differential equation is not homogeneous due to the e^{-2x} terms. Since it can be written in the form $dy/dx + P(x)y = Q(x)y^n$, namely,

$$\frac{dy}{dx} - y = e^{2x}y^3.$$

it is a Bernoulli equation. The differential equation does not have linear coefficients, and it is not of the form $y' = G(ax + by)$ either.

8. Here, the variables are y and θ. Writing

$$\frac{dy}{d\theta} = -\frac{y^3 - \theta y^2}{2\theta^2 y} = -\frac{(y/\theta)^3 - (y/\theta)^2}{2(y/\theta)},$$

we see that the right-hand side is a function of y/θ alone. Hence, the equation is homogeneous.

10. First, we write the equation in the form

$$\frac{dy}{dx} = \frac{-3x^2 + y^2}{xy - x^3y^{-1}} = \frac{y^3 - 3x^2y}{xy^2 - x^3} = \frac{(y/x)^3 - 3(y/x)}{(y/x)^2 - 1}.$$

Therefore, it is homogeneous, and we we make a substitution $y/x = u$ or $y = xu$. Then $y' = u + xu'$, and the equation becomes

$$u + x\frac{du}{dx} = \frac{u^3 - 3u}{u^2 - 1}.$$

Separating variables and integrating yields

$$x\frac{du}{dx} = \frac{u^3 - 3u}{u^2 - 1} - u = -\frac{2u}{u^2 - 1} \qquad \Rightarrow \qquad \frac{u^2 - 1}{u}\,du = -\frac{2}{x}\,dx$$

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$$\Rightarrow \quad \int \frac{u^2-1}{u}\,du = -\int \frac{2}{x}\,dx \quad \Rightarrow \quad \int \left(u-\frac{1}{u}\right)du = -2\int \frac{dx}{x}$$
$$\Rightarrow \quad \frac{1}{2}u^2 - \ln|u| = -2\ln|x| + C_1 \quad \Rightarrow \quad u^2 - \ln\left(u^2\right) + \ln(x^4) = C.$$

Substituting back y/x for u and simplifying, we finally get

$$\left(\frac{y}{x}\right)^2 - \ln\left(\frac{y^2}{x^2}\right) + \ln(x^4) = C \quad \Rightarrow \quad \frac{y^2}{x^2} - \ln\left(\frac{y^2}{x^6}\right) = C.$$

12. From

$$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy} = -\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right),$$

making the substitution $v = y/x$, we obtain

$$v + x\frac{dv}{dx} = -\frac{1}{2}\left(\frac{1}{v}+v\right) = -\frac{1+v^2}{2v} \quad \Rightarrow \quad x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v = -\frac{1+3v^2}{2v}$$
$$\Rightarrow \quad \frac{2v\,dv}{1+3v^2} = -\frac{dx}{x} \quad \Rightarrow \quad \int \frac{2v\,dv}{1+3v^2} = -\int \frac{dx}{x}$$
$$\Rightarrow \quad \frac{1}{3}\ln\left(1+3v^2\right) = -\ln|x| + C_2 \quad \Rightarrow \quad 1+3v^2 = C_1|x|^{-3},$$

where $C_1 = e^{3C_2}$ is any positive constant. Making the back substitution, we finally get

$$1 + 3\left(\frac{y}{x}\right)^2 = \frac{C_1}{|x|^3} \quad \Rightarrow \quad 3\left(\frac{y}{x}\right)^2 = \frac{C_1}{|x|^3} - 1 = \frac{C_1 - |x|^3}{|x|^3}$$
$$\Rightarrow \quad 3|x|y^2 = C_1 - |x|^3 \quad \Rightarrow \quad 3|x|y^2 + |x|^3 = C_1 \quad \Rightarrow \quad 3xy^2 + x^3 = C\,,$$

where $C = \pm C_1$ is any nonzero constant.

14. Substituting $v = y/\theta$ yields

$$v + \theta\frac{dv}{d\theta} = \sec v + v \quad \Rightarrow \quad \theta\frac{dv}{d\theta} = \sec v$$
$$\Rightarrow \quad \cos v\,dv = \frac{d\theta}{\theta} \quad \Rightarrow \quad \int \cos v\,dv = \int \frac{d\theta}{\theta}$$
$$\Rightarrow \quad \sin v = \ln|\theta| + C \quad \Rightarrow \quad y = \theta\arcsin\left(\ln|\theta| + C\right).$$

16. We rewrite the equation in the form

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\frac{y}{x} + 1\right)$$

and substitute $v = y/x$ to get

$$v + x\frac{dv}{dx} = v\left(\ln v + 1\right) \quad \Rightarrow \quad x\frac{dv}{dx} = v\ln v \quad \Rightarrow \quad \int \frac{dv}{v\ln v} = \int \frac{dx}{x}$$

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$$\Rightarrow \quad \ln|\ln v| = \ln|x| + C_1 \quad \Rightarrow \quad \ln v = \pm e^{C_1}x = Cx \quad \Rightarrow \quad v = e^{Cx},$$

where $C \neq 0$ is any constant. Note that, separating variables, we lost a solution, $v \equiv 1$, which can be included in the above formula by letting $C = 0$. Thus we have $v = e^{Cx}$. where C is any constant. Substituting back $y = xv$ yields a general solution

$$y = xe^{Cx}$$

to the given equation.

18. With $z = x + y + 2$ and $z' = 1 + y'$, we have

$$\frac{dz}{dx} = z^2 + 1 \quad \Rightarrow \quad \frac{dz}{z^2+1} = dx \quad \Rightarrow \quad \int \frac{dz}{z^2+1} = \int dx$$
$$\Rightarrow \quad \arctan z = x + C \quad \Rightarrow \quad x + y + 2 = z = \tan(x + C)$$
$$\Rightarrow \quad y = \tan(x + C) - x - 2.$$

20. Substitution $z = x - y$ yields

$$1 - \frac{dz}{dx} = \sin z \quad \Rightarrow \quad \frac{dz}{dx} = 1 - \sin z \quad \Rightarrow \quad \frac{dz}{1 - \sin z} = dx$$
$$\Rightarrow \quad \int \frac{dz}{1 - \sin z} = \int dx = x + C.$$

The left-hand side integral can be found as follows.

$$\begin{aligned} \int \frac{dz}{1 - \sin z} &= \int \frac{(1 + \sin z)dz}{1 - \sin^2 z} = \int \frac{(1 + \sin z)dz}{\cos^2 z} \\ &= \int \sec^2 z + \int \tan z \sec z \, dz = \tan z + \sec z. \end{aligned}$$

Thus, a general solution is given implicitly by

$$\tan(x - y) + \sec(x - y) = x + C.$$

22. Dividing the equation by y^3 yields

$$y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}.$$

We now make a substitution $v = y^{-2}$ so that $v' = -2y^{-3}y'$, and get

$$\frac{dv}{dx} + 2v = -2e^{2x}.$$

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This is a linear equation. So,

$$\mu(x) = \exp\left(\int 2dx\right) = e^{2x},$$
$$v(x) = e^{-2x}\int\left(-2e^{2x}\right)e^{2x}dx = -(1/2)e^{-2x}\left(e^{4x}+C\right) = -\frac{e^{2x}+Ce^{-2x}}{2}.$$

Therefore,

$$\frac{1}{y^2} = -\frac{e^{2x}+Ce^{-2x}}{2} \qquad \Rightarrow \qquad y = \pm\sqrt{-\frac{2}{e^{2x}+Ce^{-2x}}}.$$

Dividing the equation by y^3, we lost a constant solution $y \equiv 0$.

24. We divide this Bernoulli equation by $y^{1/2}$ and make a substitution $v = y^{1/2}$.

$$y^{-1/2}\frac{dy}{dx} + \frac{1}{x-2}y^{1/2} = 5(x-2)$$
$$\Rightarrow \qquad 2\frac{dv}{dx} + \frac{1}{x-2}v = 5(x-2) \qquad \Rightarrow \qquad \frac{dv}{dx} + \frac{1}{2(x-2)}v = \frac{5(x-2)}{2}.$$

An integrating factor for this linear equation is

$$\mu(x) = \exp\left[\int\frac{dx}{2(x-2)}\right] = \sqrt{|x-2|}.$$

Therefore,

$$\begin{aligned} v(x) &= \frac{1}{\sqrt{|x-2|}}\int\frac{5(x-2)\sqrt{|x-2|}}{2}dx \\ &= \frac{1}{\sqrt{|x-2|}}\left(|x-2|^{5/2}+C\right) = (x-2)^2 + C|x-2|^{-1/2}. \end{aligned}$$

Since $y = v^2$, we finally get

$$y = \left[(x-2)^2 + C|x-2|^{-1/2}\right]^2.$$

In addition, $y \equiv 0$ is a (lost) solution.

26. Multiplying the equation by y^2, we get

$$y^2\frac{dy}{dx} + y^3 = e^x.$$

With $v = y^3$, $v' = 3y^2y'$, the equation becomes

$$\frac{1}{3}\frac{dv}{dx} + v = e^x \qquad \Rightarrow \qquad \frac{dv}{dx} + 3v = 3e^x \qquad \Rightarrow \qquad \frac{d}{dx}\left(e^{3x}v\right) = 3e^{4x}$$

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Integrating yields

$$e^{-x}y = \int e^{2x}dx = \frac{1}{2}e^{2x} + C$$
$$\Rightarrow \qquad y = \left(\frac{1}{2}e^{2x} + C\right)e^{x} = \frac{e^{3x}}{2} + Ce^{x}.$$

10. This is a linear equation with the dependent variable r and independent variable θ. The method we will use to solve this equation is exactly the same as the method we use to solve an equation in the variables x and y. Thus, we have $P(\theta) = \tan\theta$ and $Q(\theta) = \sec\theta$, which are continuous on any interval not containing odd multiples of $\pi/2$. We proceed as usual to find the integrating factor $\mu(\theta)$.

$$\mu(\theta) = \exp\left(\int \tan\theta\, d\theta\right) = e^{-\ln|\cos\theta|} = \frac{1}{|\cos\theta|} = |\sec\theta|\,.$$

Note that, if $\mu(\theta)$ is an integrating factor, then $-\mu(\theta)$ is an integrating factor as well. Thus, we can take $\mu(\theta) = \sec\theta$. Multiplying the equation by $\mu(\theta)$ yields

$$\sec\theta\frac{dr}{d\theta} + (\sec\theta\tan\theta)r = \sec^2\theta \qquad \Rightarrow \qquad \frac{d}{d\theta}(r\sec\theta) = \sec^2\theta\,.$$

Integrating with respect to θ yields

$$r\sec\theta = \int \sec^2\theta\, d\theta = \tan\theta + C \quad \Rightarrow \quad r = \cos\theta\tan\theta + C\cos\theta \quad \Rightarrow \quad r = \sin\theta + C\cos\theta.$$

Because of the continuity of $P(\theta)$ and $Q(\theta)$, this solution is valid on any open interval that whose endpoints are consecutive odd multiples of $\pi/2$.

12. Here, $P(x) = 4$, $Q(x) = x^2e^{-4x}$. So, $\mu(x) = e^{4x}$ and

$$\frac{d}{dx}\left(e^{4x}y\right) = x^2 \qquad \Rightarrow \qquad y = e^{-4x}\int x^2dx = e^{-4x}\left(\frac{x^3}{3} + C\right).$$

14. In standard form, we have

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^2} - 3x\,.$$

Therefore, $P(x) = 3/x$,

$$\mu(x) = \exp\left(\int \frac{3\,dx}{x}\right) = x^3$$

and

$$y = \frac{1}{x^3}\int\left(x\sin x - 3x^4\right)dx = \frac{1}{x^3}\left(\sin x - x\cos x - \frac{3}{5}x^5 + C\right).$$

(Integration by parts was used to integrate the first term.)

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$$\Rightarrow \qquad v = e^{-3x}\int 3e^{4x}dx = \frac{3e^{x}}{4} + Ce^{-3x}.$$

Therefore,

$$y = \sqrt[3]{v} = \sqrt[3]{\frac{3e^{x}}{4} + Ce^{-3x}}\,.$$

28. First, we note that $y \equiv 0$ is a solution, which will be lost when we divide the equation by y^3 and make a substitution $v = y^{-2}$ to get a linear equation

$$y^{-3}\frac{dy}{dx} + y^{-2} + x = 0 \qquad \Rightarrow \qquad \frac{dv}{dx} - 2v = 2x.$$

An integrating factor for this equation is

$$\mu(x) = \exp\left[\int(-2)dx\right] = e^{-2x}.$$

Thus,

$$\begin{aligned} v &= e^{2x}\int 2xe^{-2x}dx = e^{2x}\left(-xe^{-2x} + \int e^{-2x}dx\right) \\ &= e^{2x}\left(-xe^{-2x} - \frac{e^{-2x}}{2} + C\right) = -x - \frac{1}{2} + Ce^{2x}. \end{aligned}$$

So,

$$y = \pm v^{-1/2} = \pm\frac{1}{\sqrt{-x - \frac{1}{2} + Ce^{2x}}}\,.$$

30. We make a substitution

$$x = u + h, \quad y = v + k,$$

where h and k satisfy the system (14) in the text, i.e.,

$$\begin{cases} h + k - 1 = 0 \\ k - h - 5 = 0. \end{cases}$$

Solving yields $h = -2$, $k = 3$. Thus, $x = u - 2$ and $y = v + 3$. Since $dx = du$, $dy = dv$, this substitution leads to the equation

$$(u + v)du + (v - u)dv = 0 \qquad \Rightarrow \qquad \frac{dv}{du} = \frac{u + v}{u - v} = \frac{1 + (v/u)}{1 - (v/u)}\,.$$

This is a homogeneous equation, and a substitution $z = v/u$ ($v' = z + uz'$) yields

$$z + u\frac{dz}{du} = \frac{1 + z}{1 - z} \qquad \Rightarrow \qquad u\frac{dz}{du} = \frac{1 + z}{1 - z} - z = \frac{1 + z^2}{1 - z}$$

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$$\begin{aligned}
&\Rightarrow \quad \frac{(1-z)dz}{1+z^2} = \frac{du}{u} \\
&\Rightarrow \quad \arctan z - \frac{1}{2}\ln\left(1+z^2\right) = \ln|u| + C_1 \\
&\Rightarrow \quad 2\arctan\frac{v}{u} - \ln\left[u^2\left(1+z^2\right)\right] = 2C_1 \\
&\Rightarrow \quad 2\arctan\frac{v}{u} - \ln\left(u^2+v^2\right) = C.
\end{aligned}$$

The back substitution yields

$$2\arctan\left(\frac{y-3}{x+2}\right) - \ln\left[(x+2)^2 + (y-3)^2\right] = C.$$

32. To obtain a homogeneous equation, we make a substitution $x = u + h$, $y = v + k$ with h and k satisfying

$$\begin{cases} 2h + k + 4 = 0 \\ h - 2k - 2 = 0 \end{cases} \quad \Rightarrow \quad h = -\frac{6}{5}, \quad k = -\frac{8}{5}.$$

This substitution yields

$$(2u+v)du + (u-2v)dv = 0 \quad \Rightarrow \quad \frac{dv}{du} = \frac{v+2u}{2v-u} = \frac{(v/u)+2}{2(v/u)-1}.$$

We now let $z = v/u$ (so, $v' = z + uz'$) and conclude that

$$\begin{aligned}
& z + u\frac{dz}{du} = \frac{z+2}{2z-1} \quad \Rightarrow \quad u\frac{dz}{du} = \frac{z+2}{2z-1} - z = \frac{-2z^2+2z+2}{2z-1} \\
&\Rightarrow \quad \int \frac{(2z-1)dz}{z^2-z-1} = -2\int\frac{du}{u} \\
&\Rightarrow \quad \ln\left|z^2-z-1\right| = -2\ln|u| + C_1 \quad \Rightarrow \quad \ln\left|u^2z^2 - u^2z - u^2\right| = C_1 \\
&\Rightarrow \quad \ln\left|v^2 - uv - u^2\right| = C_1 \\
&\Rightarrow \quad \ln\left|\left(y+\frac{8}{5}\right)^2 - \left(y+\frac{8}{5}\right)\left(x+\frac{6}{5}\right) - \left(x+\frac{6}{5}\right)^2\right| = C_1 \\
&\Rightarrow \quad (5y+8)^2 - (5y+8)(5x+6) - (5x+6)^2 = C,
\end{aligned}$$

where $C = \pm 25e^{C_1} \neq 0$ is any constant.

Separating variables, we lost two constant solutions $z(u)$, which are the zeros of the polynomial $z^2 - z - 1$. They can be included in the above formula by taking $C = 0$. Therefore, a general solution is given by

$$(5y+8)^2 - (5y+8)(5x+6) - (5x+6)^2 = C,$$

where C is an arbitrary constant.

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34. In Problem 2, we found that the given equation is of the form $dy/dx = G(y-4x)$ with $G(u) = (u-1)^2$. Thus we make a substitution $u = y - 4x$ to get

$$\frac{dy}{dx} = (y-4x-1)^2 \quad \Rightarrow \quad 4 + \frac{du}{dx} = (u-1)^2$$
$$\Rightarrow \quad \frac{du}{dx} = (u-1)^2 - 4 = (u-3)(u+1) \quad \Rightarrow \quad \int \frac{du}{(u-3)(u+1)} = \int dx\,.$$

To integrate the left-hand side, we use partial fractions decomposition,

$$\frac{1}{(u-3)(u+1)} = \frac{1}{4}\left(\frac{1}{u-3} - \frac{1}{u+1}\right).$$

Thus,

$$\frac{1}{4}\left(\ln|u-3| - \ln|u+1|\right) = x + C_1 \quad \Rightarrow \quad \ln\left|\frac{u-3}{u+1}\right| = 4x + C_2$$
$$\Rightarrow \quad \frac{u-3}{u+1} = Ce^{4x} \quad \Rightarrow \quad u = \frac{Ce^{4x}+3}{1-Ce^{4x}}$$
$$\Rightarrow \quad y = 4x + \frac{Ce^{4x}+3}{1-Ce^{4x}}\,, \tag{2.7}$$

where $C \neq 0$ is an arbitrary constant. Separating variables, we lost the constant solutions $u \equiv 3$ and $u \equiv -1$, that is, $y = 4x+3$ and $y = 4x-1$. While $y = 4x+3$ can be obtained from (2.7) by setting $C = 0$, the solution $y = 4x - 1$ is not included in (2.7). Therefore, a general solution to the given equation is

$$y = 4x + \frac{Ce^{4x}+3}{1-Ce^{4x}} \quad \text{and} \quad y = 4x - 1.$$

36. This equation has linear coefficients. Thus we make a substitution $t = u+h$ and $x = v+k$ with h and k satisfying

$$\begin{aligned} h + k + 2 &= 0, \\ 3h - k - 6 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} h &= 1, \\ k &= -3. \end{aligned}$$

As $dt = du$ and $dx = dv$, the substitution yields

$$(u+v)dv + (3u-v)du = 0 \quad \Rightarrow \quad \frac{du}{dv} = -\frac{u+v}{3u-v} = -\frac{(u/v)+1}{3(u/v)-1}\,.$$

With $z = u/v$, we have $u = vz$, $u' = z + vz'$, and the equation becomes

$$z + v\frac{dz}{dv} = -\frac{z+1}{3z-1} \quad \Rightarrow \quad v\frac{dz}{dv} = -\frac{3z^2+1}{3z-1}$$
$$\Rightarrow \quad \frac{3z-1}{3z^2+1}\,dz = -\frac{1}{v}\,dv \quad \Rightarrow \quad \int \frac{3z-1}{3z^2+1}\,dz = -\int \frac{1}{v}\,dv$$

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$$\Rightarrow \qquad \int \frac{3z\,dz}{3z^2+1} - \int \frac{dz}{3z^2+1} = -\ln|v| + C_1$$

$$\Rightarrow \qquad \frac{1}{2}\ln\left(3z^2+1\right) - \frac{1}{\sqrt{3}}\arctan\left(z\sqrt{3}\right) = -\ln|v| + C_1$$

$$\Rightarrow \qquad \ln\left[(3z^2+1)v^2\right] - \frac{2}{\sqrt{3}}\arctan\left(z\sqrt{3}\right) = C_2.$$

Making back substitution, after some algebra we get

$$\ln\left[(x+3)^2 + 3(t-1)^2\right] + \frac{2}{\sqrt{3}}\arctan\left[\frac{x+3}{\sqrt{3}(t-1)}\right] = C.$$

38. In Problem 6, we have written the equation in the form

$$\frac{dy}{dx} - y = e^{2x}y^3 \qquad \Rightarrow \qquad y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}.$$

Making a substitution $u = y^{-2}$ (and so $u' = -2y^{-3}y'$) in this Bernoulli equation, we get

$$\frac{du}{dx} + 2u = -2e^{2x} \qquad \Rightarrow \qquad \mu(x) = \exp\left(\int 2dx\right) = e^{2x}$$

$$\Rightarrow \quad \frac{d\left(e^{2x}u\right)}{dx} = -2e^{2x}e^{2x} = -2e^{4x} \qquad \Rightarrow \qquad e^{2x}u = \int\left(-2e^{4x}\right)dx = -\frac{1}{2}e^{4x} + C$$

$$\Rightarrow \quad u = -\frac{1}{2}e^{2x} + Ce^{-2x} \qquad \Rightarrow \qquad y^{-2} = -\frac{1}{2}e^{2x} + Ce^{-2x}.$$

Also, we lost the constant solution $y \equiv 0$ when divided the equation by y^3.

40. Since the equation is homogeneous, we make a substitution $u = y/\theta$. Thus, we get

$$\frac{dy}{d\theta} = -\frac{(y/\theta)^3 - (y/\theta)^2}{2(y/\theta)} \qquad \Rightarrow \qquad u + \theta\frac{du}{d\theta} = -\frac{u^3 - u^2}{2u} = -\frac{u^2 - u}{2}$$

$$\Rightarrow \qquad \theta\frac{du}{d\theta} = -\frac{u^2+u}{2} \qquad \Rightarrow \qquad \frac{2du}{u(u+1)} = -\frac{d\theta}{\theta}$$

$$\Rightarrow \qquad \int\frac{2du}{u(u+1)} = -\int\frac{d\theta}{\theta} \qquad \Rightarrow \qquad \ln\frac{u^2}{(u+1)^2} = -\ln|\theta| + C_1,$$

which gives

$$\frac{u^2}{(u+1)^2} = \frac{C}{\theta}, \qquad C \neq 0.$$

Back substitution $u = y/\theta$ yields

$$\frac{y^2}{(\theta+y)^2} = \frac{C}{\theta} \qquad \Rightarrow \qquad \theta y^2 = C(\theta+y)^2, \qquad C \neq 0.$$

When $C = 0$, the above formula gives $\theta \equiv 0$ or $y \equiv 0$, which were lost in separating variables. Also, we lost another solution, $u + 1 \equiv 0$ or $y = -\theta$. Thus, the answer is

$$\theta y^2 = C(\theta+y)^2 \quad \text{and} \quad y = -\theta,$$

where C is an arbitrary constant.

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42. Suggested substitution, $y = vx^2$ (so that $y' = 2xv + x^2v'$), yields

$$2xv + x^2\frac{dv}{dx} = 2vx + \cos v \qquad \Rightarrow \qquad x^2\frac{dv}{dx} = \cos v.$$

Solving this separable equation, we obtain

$$\begin{aligned} \frac{dv}{\cos v} = \frac{dx}{x^2} \qquad &\Rightarrow \qquad \ln|\sec v + \tan v| = -x^{-1} + C_1 \\ \Rightarrow \qquad &\sec v + \tan v = \pm e^{C_1}e^{-1/x} = Ce^{-1/x} \\ \Rightarrow \qquad &\sec\left(\frac{y}{x^2}\right) + \tan\left(\frac{y}{x^2}\right) = Ce^{-1/x}, \end{aligned}$$

where $C = \pm e^{C_1}$ is an arbitrary nonzero constant. With $C = 0$, this formula also includes lost solutions

$$y = \left[\frac{\pi}{2} + (2k+1)\pi\right]x^2, \quad k = 0, \pm 1, \pm 2, \ldots .$$

So, together with the other set of lost solutions,

$$y = \left(\frac{\pi}{2} + 2k\pi\right)x^2, \quad k = 0, \pm 1, \pm 2, \ldots ,$$

we get a general solution to the given equation.

44. From

$$\frac{dy}{dx} = -\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2},$$

using that $a_2 = ka_1$ and $b_2 = kb_1$, we obtain

$$\frac{dy}{dx} = -\frac{a_1x + b_1y + c_1}{ka_1x + kb_1y + c_2} = -\frac{a_1x + b_1y + c_1}{k(a_1x + b_1y) + c_2} = G\left(a_1x + b_1y\right),$$

where

$$G(t) = -\frac{t + c_1}{kt + c_2}.$$

46. **(a)** Substituting $y = u + 1/v$ into the Riccati equation (18) and using the fact that $u(x)$ is a solution, we obtain

$$\begin{aligned} &\frac{d}{dx}\left(u + v^{-1}\right) = P(x)\left(u + v^{-1}\right)^2 + Q(x)\left(u + v^{-1}\right) + R(x) \\ \Leftrightarrow \quad &\frac{du}{dx} - v^{-2}\frac{dv}{dx} = P(x)u^2 + 2P(x)u(x)v^{-1} + P(x)v^{-2} \\ &\qquad\qquad + Q(x)u + Q(x)v^{-1} + R(x) \\ \Leftrightarrow \quad &\frac{du}{dx} - v^{-2}\frac{dv}{dx} = \left[P(x)u^2 + Q(x)u + R(x)\right] \end{aligned}$$

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$$+\left[2P(x)u(x)+Q(x)\right]v^{-1}+P(x)v^{-2}$$

$$\Leftrightarrow \quad -v^{-2}\frac{dv}{dx}=\left[2P(x)u(x)+Q(x)\right]v^{-1}+P(x)v^{-2}$$

$$\Leftrightarrow \quad \frac{dv}{dx}+\left[2P(x)u(x)+Q(x)\right]v=-P(x),$$

which is indeed a linear equation with respect to v.

(b) Writing

$$\frac{dy}{dx}=x^3y^2+\left(-2x^4+\frac{1}{x}\right)y+x^5$$

and using notations in (18), we see that $P(x)=x^3$, $Q(x)=(-2x^4+1/x)$, and $R(x)=x^5$. So, using part (a), we are looking for other solutions to the given equation of the form $y=x+1/v$, where $v(x)$ satisfies

$$\frac{dv}{dx}+\left[2\left(x^3\right)x+\left(-2x^4+\frac{1}{x}\right)\right]v=\frac{dv}{dx}+-\frac{1}{x}v=-x^3.$$

Since an integrating factor for this linear equation is

$$\mu(x)=\exp\left(\int\frac{dx}{x}\right)=x,$$

we obtain

$$v=x^{-1}\int\left(-x^4\right)dx=\frac{-x^5+C}{5x} \qquad \Rightarrow \qquad \frac{1}{v}=\frac{5x}{C-x^5},$$

and so a general solution is given by $y=x+(5x)/\left(C-x^5\right)$.

REVIEW PROBLEMS

2. $y=-8x^2-4x-1+Ce^{4x}$

4. $\frac{x^3}{6}-\frac{4x^2}{5}+\frac{3x}{4}-Cx^{-3}$

6. $y^{-2}=2\ln|1-x^2|+C$ and $y\equiv 0$

8. $y=\left(Cx^2-2x^3\right)^{-1}$ and $y\equiv 0$

10. $x+y+2y^{1/2}+\arctan(x+y)=C$

12. $2ye^{2x}+y^3e^x=C$

14. $x=\frac{t^2(t-1)}{2}+t(t-1)+3(t-1)\ln|t-1|+C(t-1)$

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16. $y = \cos x \ln|\cos x| + C\cos x$

18. $y = 1 - 2x + \sqrt{2}\tan\left(\sqrt{2}\,x + C\right)$

20. $y = \left(C\theta^{-3} - \dfrac{12\theta^2}{5}\right)^{1/3}$

22. $(3y - 2x + 9)(y + x - 2)^4 = C$

24. $2\sqrt{xy} + \sin x - \cos y = C$

26. $y = Ce^{-x^2/2}$

28. $(y+3)^2 + 2(y+3)(x+2) - (x+2)^2 = C$

30. $y = Ce^{4x} - x - \dfrac{1}{4}$

32. $y^2 = x^2\ln(x^2) + 16x^2$

34. $y = x^2\sin x + \dfrac{2x^2}{\pi^2}$

36. $\sin(2x + y) - \dfrac{x^3}{3} + e^y = \sin 2 + \dfrac{2}{3}$

38. $y = \left[2 - \left(\dfrac{1}{4}\right)\arctan\left(\dfrac{x}{2}\right)\right]^2$

40. $y = \dfrac{8}{1 - 3e^{-4x} - 4x}$

TABLES

n	x_n	y_n	n	x_n	y_n
1	0.1	1.475	6	0.6	1.353368921
2	0.2	1.4500625	7	0.7	1.330518988
3	0.3	1.425311875	8	0.8	1.308391369
4	0.4	1.400869707	9	0.9	1.287062756
5	0.5	1.376852388	10	1.0	1.266596983

Table 2–A: Euler's approximations to $y' = x - y$, $y(0) = 0$, on $[0, 1]$ with $h = 0.1$.

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FIGURES

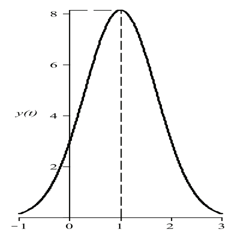

Figure 2–A: The graph of the solution in Problem 28.

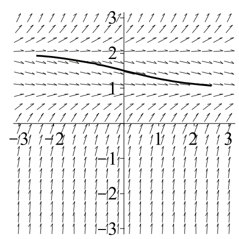

Figure 2–B: The direction field and solution curve in Problem 32.

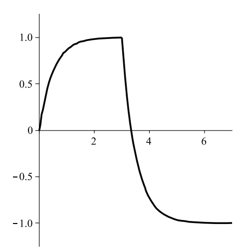

Figure 2–C: The graph of the solution in Problem 32.

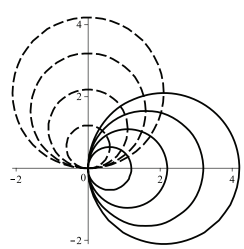

Figure 2–D: Curves and their orthogonal trajectories in Problem 34.

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