## CHAPTER 1: SYSTEMS OF LINEAR EQUATIONS AND MATRICES

1. Which of the following are linear equations in $x_1, x_2$, and $x_3$?

a) $3x_1 - x_2 + 5x_3 = 4$

b) $x_1 - 4x_2x_3 = 3$

c) $x_1^2 + 2x_2 - 3x_3 = 0$

d) $\sqrt{3}x_1 - \sqrt{2}x_2 + x_3 = 5$

e) $\pi x_1 + \pi^2 x_2 = \pi^3 x_3$

f) $\sqrt{5}x_1 + 5\sqrt{x_2} - x_3 = 1$

In questions 2–5, find the augmented matrix for the system of linear equations.

2.
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_1 + 3x_3 &= 2 \\ 2x_1 + x_2 &= 5 \end{aligned}$$

3.
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 7 \\ x_1 - x_2 + x_3 - x_4 &= 12 \end{aligned}$$

4.
$$\begin{aligned} \sqrt{2}x_1 + \sqrt{6}x_2 &= 2 \\ \sqrt{3}x_1 + \sqrt{2}x_2 + \sqrt{6}x_3 &= 3 \\ \sqrt{3}x_2 + \sqrt{2}x_3 &= 6 \end{aligned}$$

5.
$$\begin{aligned} x_1 - x_2 &= 5 \\ x_1 + x_2 &= -1 \\ 2x_1 + 3x_2 &= 4 \end{aligned}$$

In questions 6–9, find a system of linear equations corresponding to the augmented matrix.

6. $$\begin{bmatrix} 1 & 11 & 6 & 3 \\ 9 & 4 & 0 & -2 \\ 5 & 9 & -4 & 1 \end{bmatrix}$$

7. $$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{bmatrix}$$

8. $$\begin{bmatrix} 1/3 & 1/4 & 1/8 \\ 1/6 & 1/2 & 1/8 \\ 1/2 & 1/4 & 3/4 \end{bmatrix}$$

9. $$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 4 \\ 1 & 1 & 0 & 1 & 1 & 9 \end{bmatrix}$$

10. In each part, determine whether the matrix is in row–echelon form, reduced row–echelon form, both, or neither.

a) $$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -7 & 5 \\ 0 & 0 & 1 & 14 \end{bmatrix}$$

c) $$\begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

d) $$\begin{bmatrix} 1 & 0 & 0 & 11 & -3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

e) $$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

f) $$\begin{bmatrix} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 16 \\ 0 & 0 & 0 & 1 & 20 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In questions 11–16, solve the system by Gaussian elimination or by Gauss–Jordan elimination.

11.
$$\begin{aligned} x - y - z &= 0 \\ 2x + y + z &= 3 \\ 3x - z &= 0 \end{aligned}$$

12.
$$\begin{aligned} x_1 - x_2 - 5x_3 &= -1 \\ -2x_1 + 2x_2 + 11x_3 &= 1 \\ 3x_1 - x_2 + x_3 &= 3 \end{aligned}$$

13.
$$\begin{aligned} x_1 + x_2 + 3x_3 &= a \\ x_1 + 2x_3 &= b \\ 2x_1 - x_2 + x_3 &= c \end{aligned}$$

14.
$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &= 4 \\ 3x_1 + 5x_2 + x_3 &= -8 \\ -4x_1 - 7x_2 - 2x_3 &= 13 \end{aligned}$$

15.
$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & & & + & 2x_4 & = & 2 \\ 2x_1 & + & x_2 & + & 5x_3 & + & 4x_4 & = & -16 \\ 2x_1 & + & 3x_2 & + & 3x_3 & & & = & 4 \\ 3x_1 & + & 11x_2 & - & 2x_3 & + & 11x_4 & = & -1 \end{array}$$

16.
$$\begin{array}{rcrcr} x & - & y & = & 2 \\ 2x & + & y & = & 1 \\ x & + & 3y & = & -1 \end{array}$$

In questions 17–20, solve the system by Gaussian elimination or by Gauss–Jordan elimination

17.
$$\begin{array}{rcrcrcr} x & - & 4y & + & z & = & 0 \\ 2x & - & 3y & + & 7z & = & 0 \\ x & - & 2y & & & = & 0 \end{array}$$

18.
$$\begin{array}{rcrcrcr} x_1 & + & 7x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 14x_2 & + & 5x_3 & = & 0 \\ 3x_1 & + & 21x_2 & + & 5x_3 & = & 0 \end{array}$$

19.
$$\begin{array}{rcrcrcr} 2x_1 & + & 6x_2 & - & 4x_3 & = & 0 \\ 3x_1 & + & 9x_2 & - & 6x_3 & = & 0 \\ -4x_1 & - & 12x_2 & + & 8x_3 & = & 0 \end{array}$$

20.
$$\begin{array}{rcrcrcrcr} w & + & 6x & - & y & + & 2z & = & 0 \\ -w & - & 4x & + & 3y & + & 8z & = & 0 \\ 2w & + & 11x & & & + & 5z & = & 0 \\ 2w & + & 14x & - & 2y & + & 13z & = & 0 \end{array}$$

21. Consider the matrices:

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} , \quad B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} , \quad C = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} , \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Compute the following.

a) $B - A$ b) $3B + 4C - 2A$ c) $AB - BA$ d) $BAI - IBC$
e) $(A - 2I)^2$ f) $(A^2 - B^2)C^2$ g) $A^{-1} + AA^{-1}$ h) $BC - (C^{-1}B^{-1})^{-1}$
i) $CC^T$ j) $(3AB)^T - B^TA^T$ k) $tr(A)$ l) $tr(3A^T - 4B^TC^T)$

22. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 5 \\ 1 & 6 & 0 \end{bmatrix} , \quad B = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 1 \end{bmatrix} , \quad C = \begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 5 & 0 \end{bmatrix}$$

Compute the following.

a) $BC$ b) $CB$ c) $A^2 - 2CB$
d) $(AC + 3B^T)^T$ e) $BCB$ f) $tr(1/3BAC)$

23. Consider the matrices and scalars:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} , \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} , \quad C = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix} , \quad a = 2 , \quad b = 3$$

Use these to show the following:

a) $A + (B + C) = (A + B) + C$ b) $(AB)C = A(BC)$
c) $A(B + C) = AB + AC$ d) $a(bC) = (ab)C$
e) $(A + B)^T = A^T + B^T$ f) $(AB)^T = B^TA^T$

In questions 24–25, find matrices $A$, $X$, and $B$ such that the given system of linear equations can be abbreviated $AX = B$.

24. $$\begin{aligned} x + 2y - z &= 5 \\ 2x - y + 11z &= 19 \\ 3x + y + z &= 11 \end{aligned}$$

25. $$\begin{aligned} - x_2 - x_3 - x_4 &= 1 \\ -x_1 - x_3 - x_4 &= 1 \\ -x_1 - x_2 - x_4 &= 1 \\ -x_1 - x_2 - x_3 &= 1 \end{aligned}$$

In questions 26–27, express the matrix equation as a system of linear equations.

26. $$\begin{bmatrix} -1 & 7 & 0 \\ 0 & 4 & 3 \\ 6 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

27. $$\begin{bmatrix} 1/3 & 1/2 & 0 & -1/6 \\ 2/3 & 1/6 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix}$$

28. For each matrix, indicate whether it is elementary and, if it is, indicate which row operation will restore the matrix to an identity matrix.

a) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ b) $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -9 & 0 & 1 \end{bmatrix}$

In questions 29–32, consider the matrices:

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 2 & 10 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 2 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Find elementary matrices $E_1$, $E_2$, $E_3$, and $E_4$ satisfying the given equation.

29. $E_1 A = B$ 30. $E_2 B = A$ 31. $E_3 B = C$ 32. $E_4 C = B$

In questions 33–40, use Gauss-Jordan elimination to find the inverse of the given matrix if the matrix is invertible.

33. $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$ 34. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ 35. $\begin{bmatrix} 1 & -5 & 3 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

36. $\begin{bmatrix} 1 & -5 & 5 \\ 4 & 3 & 2 \\ 9 & 1 & 9 \end{bmatrix}$ 37. $\begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1 & 1/4 & 1/2 \end{bmatrix}$ 38. $\begin{bmatrix} 1 & -3 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

39. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & k_1 \\ 1 & k_2 & 0 \end{bmatrix}$

40. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ k & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & k & 1 \end{bmatrix}$

In questions 41–48, use an inverse matrix to solve the system of equations.

41. $\begin{aligned} 7x + 2y &= 1 \\ 3x + y &= 5 \end{aligned}$

42. $\begin{aligned} x_1 + 4x_2 &= 3 \\ 5x_1 + x_2 &= -2 \end{aligned}$

43. $\begin{aligned} x + 6y + 5z &= 3 \\ y + z &= -1 \\ 2z &= -4 \end{aligned}$

44. $\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ -2x_1 + x_2 + x_3 &= -1 \\ -3x_1 + x_2 + 5x_3 &= -1 \end{aligned}$

45. $\begin{aligned} 2x_1 - 3x_2 + 10x_3 &= 10 \\ - x_2 + 2x_3 &= 30 \\ -4x_1 + 8x_2 - 19x_3 &= -20 \end{aligned}$

46. $\begin{aligned} x_1 - 2x_2 - 6x_3 - 2x_4 &= 4 \\ x_1 + x_2 + 2x_3 + x_4 &= -2 \\ x_1 - x_3 &= 1 \\ 3x_1 + 3x_2 + 7x_3 + 4x_4 &= 3 \end{aligned}$

47. $\begin{aligned} .3x + .4y &= b_1 \\ .7x + .6y &= b_2 \end{aligned}$

48. $\begin{aligned} x_1 + x_2 + x_3 &= b_1 \\ 3x_1 + 4x_2 + 5x_3 &= b_2 \\ -2x_1 + x_3 &= b_3 \end{aligned}$

In questions 49–52, find the conditions that the "$b$'s" must satisfy for the system to be consistent.

49. $\begin{aligned} 6x - 3y &= b_1 \\ -10x + 5y &= b_2 \end{aligned}$

50. $\begin{aligned} x_1 + 4x_2 - 3x_3 &= b_1 \\ 2x_1 + 6x_2 + 5x_3 &= b_2 \\ x_1 = 6x_2 - 14x_3 &= b_3 \end{aligned}$

51. $\begin{aligned} x_1 + 3x_2 + x_3 &= b_1 \\ 2x_1 + 7x_2 + 3x_3 &= b_2 \\ -x_1 - x_2 + 2x_3 &= b_3 \end{aligned}$

52. $\begin{aligned} x_1 + 4x_2 + 7x_3 &= b_1 \\ 2x_1 + 5x_2 + x_3 &= b_2 \\ x_1 + 7x_2 + 20x_3 &= b_3 \\ -x_1 + 2x_2 + 19x_3 &= b_4 \end{aligned}$

## WRITING QUESTIONS

53. Discuss the relative merits of these three methods of solving systems of linear equations:

    a) simple elimination
    b) Gaussian elimination
    c) Gauss–Jordan elimination

54. Why do we study row operations as opposed to column operations on a matrix?

55. In Gauss and Gauss–Jordan elimination, why is it that 0's and 1's established in one column are not disturbed by the pivoting done in columns further to the right?

56. Describe what you have observed as to the effect of the number of equations and the number of variables upon the size of the solution set of a linear system.

57. Describe some ways that multiplication of matrices has properties different from multiplication of real numbers.

58. What would be wrong with defining matrix multiplication for matrices of the same size by multiplying them entry–by–entry, as with addition and subtraction?

59. Why does a multiplicative identity matrix not consist entirely of 1's?

60. Why is it impossible for a linear system to have exactly two solutions? Explain it graphically so that a high school student could understand.

61. Why are multiplicative inverses of matrices so important in linear algebra?

62. Why is the method of solving a system of linear equations by using the inverse of the coefficient matrix not a general method for solving such systems?

# CHAPTER 1: SOLUTIONS

## SECTION 1.1 *INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS*

1. a) yes b) no c) no d) yes e) yes f) no

2. $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 1 & -1 & 1 & -1 & 12 \end{bmatrix}$

4. $\begin{bmatrix} \sqrt{2} & \sqrt{6} & 0 & 2 \\ \sqrt{3} & \sqrt{2} & \sqrt{6} & 3 \\ 0 & \sqrt{3} & \sqrt{2} & 6 \end{bmatrix}$

5. $\begin{bmatrix} 1 & -1 & 5 \\ 1 & 1 & -1 \\ 2 & 3 & 4 \end{bmatrix}$

6. $\begin{aligned} x_1 + 11x_2 + 6x_3 &= 3 \\ 9x_1 + 4x_2 \phantom{+ 6x_3} &= -2 \\ 5x_1 + 9x_2 - 4x_3 &= 1 \end{aligned}$

7. $\begin{aligned} x_1 \phantom{+x_2+x_3} &= 1 \\ x_2 &= 5 \\ x_3 &= 10 \end{aligned}$

8. $\begin{aligned} 1/3x_1 + 1/4x_2 &= 1/8 \\ 1/6x_1 + 1/2x_2 &= 1/8 \\ 1/2x_1 + 1/4x_2 &= 3/4 \end{aligned}$

9. $\begin{aligned} x_1 - x_2 + x_3 - x_4 + x_5 &= 0 \\ x_1 \phantom{- x_2} + x_3 \phantom{- x_4} - x_5 &= 4 \\ x_1 + x_2 + \phantom{x_3} + x_4 + x_5 &= 9 \end{aligned}$

## SECTION 1.2 *GAUSSIAN ELIMINATION*

10. a) neither b) row–echelon c) neither

d) row–echelon e) neither f) both

11. $\begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 1 & 1 & 3 \\ 3 & 0 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & 3 & 3 \\ 3 & 0 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & 3 & 3 \\ 0 & 3 & 2 & 0 \end{bmatrix} \longrightarrow$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

To finish by Gaussian,

$z = 3$

$y + z = 1$ so $y = 1 - z = 1 - 3 = -2$

$x - y - z = 0$ so $x = y + z = -2 + 3 = 1$

To finish by Gauss–Jordan,

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

so $x = 1$, $y = -2$, $z = 3$.

12. $$\begin{bmatrix} 1 & -1 & -5 & -1 \\ -2 & 2 & 11 & 1 \\ 3 & -1 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -5 & -1 \\ 0 & 0 & 1 & -1 \\ 3 & -1 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -5 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & 16 & 6 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & -1 & -5 & -1 \\ 0 & 2 & 16 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -5 & -1 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

To finish by Gaussian,

$$x_3 = -1$$

$$x_2 + 8x_3 = 3 \quad \text{so} \quad x_2 = 3 - 8x_3 = 3 + 8 = 11$$

$$x_1 - x_2 - 5x_3 = -1 \quad \text{so} \quad x_1 = -1 + x_2 + 5x_3 = -1 + 11 - 5 = 5$$

To finish by Gauss–Jordan,

$$\begin{bmatrix} 1 & -1 & -5 & -1 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

so $x_1 = 5$, $x_2 = 11$, $x_3 = -1$

13.

$$\begin{bmatrix} 1 & 1 & 3 & a \\ 1 & 0 & 2 & b \\ 2 & -1 & 1 & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 3 & a \\ 0 & -1 & -1 & -a+b \\ 2 & -1 & 1 & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 3 & a \\ 0 & -1 & -1 & -a+b \\ 0 & -3 & -5 & -2a+c \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 1 & 3 & a \\ 0 & 1 & 1 & a-b \\ 0 & -3 & -5 & -2a+c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 3 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & -2 & a-3b+c \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 3 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 1 & (-1/2)a + (3/2)b - (1/2)c \end{bmatrix}$$

To finish by Gaussian,

$$x_3 = (-1/2)a + (3/2)b - (1/2)c$$

$$x_2 + x_3 = a - b \quad \text{so} \quad x_2 = a - b - x_3 = a - b - \Big((-1/2)a + (3/2)b - (1/2)c\Big)$$

$$= (3/2)a - (5/2)b + (1/2)c$$

$$x_1 + x_2 + 3x_3 = a \quad \text{so} \quad x_1 = a - x_2 - 3x_3$$

$$= a - \Big((3/2)a - (5/2)b + (1/2)c\Big) - 3\Big((-1/2)a + (3/2)b - (1/2)c\Big)$$

$$= a - 2b + c$$

To finish by Gauss–Jordan,

$$\begin{bmatrix} 1 & 1 & 3 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 1 & (-1/2)a+(3/2)b-(1/2)c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & b \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 1 & (-1/2)a+(3/2)b-(1/2)c \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & a-2b+c \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 1 & (-1/2)a+(3/2)b-(1/2)c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & a-2b+c \\ 0 & 1 & 0 & (3/2)a-(5/2)b+(1/2)c \\ 0 & 0 & 1 & (-1/2)a+(3/2)b-(1/2)c \end{bmatrix}$$

so $x_1 = a-2b+c$, $x_2 = (3/2)a-(5/2)b+(1/2)c$, $x_3 = (-1/2)a+(3/2)b-(1/2)c$

14. $$\begin{bmatrix} 2 & 2 & -2 & 4 \\ 3 & 5 & 1 & -8 \\ -4 & -7 & -2 & 13 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & 2 \\ 3 & 5 & 1 & -8 \\ -4 & -7 & -2 & 13 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 4 & -14 \\ -4 & -7 & -2 & 13 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 4 & -14 \\ 0 & -3 & -6 & 21 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -7 \\ 0 & -3 & -6 & 21 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To finish by Gaussian,

$$\begin{array}{lll} \text{Let} & x_3 = t; & \\ \text{then} & x_2 + 2x_3 = -7 \quad \text{so} & x_2 = -7 - 2x_3 = -7 - 2t \\ \text{and} & x_1 + x_2 - x_3 = 2 \quad \text{so} & x_1 = 2 - x_2 + x_3 = 2 - (-7-2t) + t = 9 + 3t \end{array}$$

To finish by Gauss–Jordan,

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -3 & 9 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{lll} \text{Let} & x_3 = t; & \\ \text{then} & x_2 + 2x_3 = -7 \quad \text{so} & x_2 = -7 - 2x_3 = -7 - 2t \\ \text{and} & x_1 - 3x_3 = 9 \quad \text{so} & x_1 = 9 + 3x_3 = 9 + 3t \end{array}$$

15. $$\begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 2 & 1 & 5 & 4 & -16 \\ 2 & 3 & 3 & 0 & 4 \\ 3 & 11 & -2 & 11 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & -5 & 5 & 0 & -20 \\ 2 & 3 & 3 & 0 & 4 \\ 3 & 11 & -2 & 11 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & -5 & 5 & 0 & -20 \\ 0 & -3 & 3 & -4 & 0 \\ 3 & 11 & -2 & 11 & -1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & -5 & 5 & 0 & -20 \\ 0 & -3 & 3 & -4 & 0 \\ 0 & 2 & -2 & 5 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & -3 & 3 & -4 & 0 \\ 0 & 2 & -2 & 5 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & -4 & 12 \\ 0 & 2 & -2 & 5 & -7 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & -4 & 12 \\ 0 & 0 & 0 & 5 & -15 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 5 & -15 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To finish by Gaussian,

$x_4 = -3$

Let $\quad x_3 = t$

then $\quad x_2 - x_3 = 4 \quad$ so $\quad x_2 = 4 + x_3 = 4 + t$

and $\quad x_1 + 3x_2 + 2x_4 = 2 \quad$ so $\quad x_1 = 2 - 3x_2 - 2x_4 = 2 - 3(4 + t) - 2(-3) = -4 - 3t$

To finish by Gauss–Jordan,

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 2 & -10 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_4 = -3$

Let $\quad x_3 = t;$

then $\quad x_2 - x_3 = \ 4 \quad$ so $\quad x_2 = 4 + x_3 = 4 + t$

and $\quad x_1 + 3x_3 = -4 \quad$ so $\quad x_1 = -4 - 3x_3 = -4 - 3t$

16.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 1 & 3 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 4 & -3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

and since the last row is an inconsistency, there are no solutions.

17. $$\begin{bmatrix} 1 & -4 & 1 & 0 \\ 2 & -3 & 7 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 5 & 5 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

To finish by Gaussian,

$x_3 = 0$

$x_2 + x_3 = 0 \quad$ so $\quad x_2 = -x_3 = 0$

$x_1 - 4x_2 + x_3 = 0 \quad$ so $\quad x_1 = 4x_2 - x_3 = 0 - 0 = 0$

To finish by Gauss–Jordan,

$$\begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

i.e., $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

18. $$\begin{bmatrix} 1 & 7 & 1 & 0 \\ 2 & 14 & 5 & 0 \\ 3 & 21 & 5 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 7 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 3 & 21 & 5 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 7 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 7 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 7 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To finish by Gaussian,

$x_3 = 0$

Let $x_2 = t$;

then $x_1 + 7x_2 + x_3 = 0$ so $x_1 = -7x_2 - x_3 = -7t - 0 = -7t$

To finish by Gauss–Jordan,

$$\begin{bmatrix} 1 & 7 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3 = 0$

Let $x_2 = t$;

then $x_1 + 7x_2 = 0$ so $x_1 = -7x_2 = -7t$

19.

$$\begin{bmatrix} 2 & 6 & -4 & 0 \\ 3 & 9 & -6 & 0 \\ -4 & -12 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -2 & 0 \\ 3 & 9 & -6 & 0 \\ -4 & -12 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & -12 & 8 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_3 = t$

and $x_2 = s$;

then $x_1 + 3x_2 - 2x_3 = 0$ so $x_1 = -3x_2 + 2x_3 = -3s + 2t$

20.

$$\begin{bmatrix} 1 & 6 & -1 & 2 & 0 \\ -1 & -4 & 3 & 8 & 0 \\ 2 & 11 & 0 & 5 & 0 \\ 2 & 14 & -2 & 13 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 6 & -1 & 2 & 0 \\ 0 & 2 & 2 & 10 & 0 \\ 2 & 11 & 0 & 5 & 0 \\ 2 & 14 & -2 & 13 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 6 & -1 & 2 & 0 \\ 0 & 2 & 2 & 10 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 2 & 14 & -2 & 13 & 0 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix}1&6&-1&2&0\\0&2&2&10&0\\0&-1&2&1&0\\0&2&0&9&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&6&-1&2&0\\0&1&1&5&0\\0&-1&2&1&0\\0&2&0&9&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&6&-1&2&0\\0&1&1&5&0\\0&0&3&6&0\\0&2&0&9&0\end{bmatrix}\longrightarrow$$

$$\begin{bmatrix}1&6&-1&2&0\\0&1&1&5&0\\0&0&3&6&0\\0&0&-2&-1&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&6&-1&2&0\\0&1&1&5&0\\0&0&1&2&0\\0&0&-2&-1&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&6&-1&2&0\\0&1&1&5&0\\0&0&1&2&0\\0&0&0&3&0\end{bmatrix}\longrightarrow$$

$$\begin{bmatrix}1&6&-1&2&0\\0&1&1&5&0\\0&0&1&2&0\\0&0&0&1&0\end{bmatrix}$$

To finish by Guassian,

$z = 0$

$y + 2z = 0$ so $y = -2z = 0$

$x + y + 5z = 0$ so $x = -y - 5z = 0 - 0 = 0$

$w + 6x - y + 2z = 0$ so $w = -6x + y - 2z = 0 + 0 - 0 = 0$

To finish by Gauss–Jordan

$$\begin{bmatrix}1&6&-1&2&0\\0&1&1&5&0\\0&0&1&2&0\\0&0&0&1&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&-7&-28&0\\0&1&1&5&0\\0&0&1&2&0\\0&0&0&1&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&0&-14&0\\0&1&1&5&0\\0&0&1&2&0\\0&0&0&1&0\end{bmatrix}\longrightarrow$$

$$\begin{bmatrix}1&0&0&-14&0\\0&1&0&3&0\\0&0&1&2&0\\0&0&0&1&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&0&0&0\\0&1&0&3&0\\0&0&1&2&0\\0&0&0&1&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&2&0\\0&0&0&1&0\end{bmatrix}\longrightarrow$$

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}$$

i.e., $w = 0$, $x = 0$, $y = 0$, $z = 0$

21. (a) $B - A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix}$

(b) $3B + 4C - 2A = 3\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} - 2\begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 3 & 6 \\ 12 & 9 \end{bmatrix} + \begin{bmatrix} 4 & -4 \\ 8 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 22 & 7 \end{bmatrix}$

(c) $AB - BA = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 7 & 9 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 9 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ -6 & -6 \end{bmatrix}$

(d) $BAI - IBC = BA - BC = B(A - C) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -1 & 11 \end{bmatrix}$

(e) $(A - 2I)^2 = \begin{bmatrix} 3-2 & 1-0 \\ -1-0 & 1-2 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(f) $(A^2 - B^2)C^2 = \left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}^2 - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}^2\right)\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}^2$
$= \left(\begin{bmatrix} 8 & 4 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix}\right)\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -20 & -17 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$
$= \begin{bmatrix} -7 & 9 \\ -14 & 54 \end{bmatrix}$

(g) $A^{-1} + AA^{-1} = A^{-1} + I = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix}$

(h) $BC - (C^{-1}B^{-1})^{-1} = BC - \left(B^{-1}\right)^{-1}\left(C^{-1}\right)^{-1} = BC - BC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(i) $CC^T = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$

(j) $(3AB)^T - B^TA^T = 3B^TA^T - B^TA^T = 2B^TA^T = 2\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = 2\begin{bmatrix} 7 & 3 \\ 9 & 1 \end{bmatrix}$

(k) $tr(A) = tr\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = 3 + 1 = 4$

(l) $tr(3A^T - 4B^TC^T) = 3tr(A^T) - 4tr(B^TC^T) = 3(4) - 4tr\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\right)$
$= 12 - 4tr\begin{bmatrix} -3 & 2 \\ -1 & 4 \end{bmatrix} = 12 - 4(1) = 8$

22. (a) $BC = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 13 & 24 \end{bmatrix}$

(b) $$CB = \begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 26 & 6 & 2 \\ -3 & 3 & -3 \\ 5 & 15 & -10 \end{bmatrix}$$

(c) $$A^2 - 2CB = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 5 \\ 1 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 5 \\ 1 & 6 & 0 \end{bmatrix} - 2 \begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 11 \\ 5 & 31 & -5 \\ 1 & -4 & 31 \end{bmatrix} - 2 \begin{bmatrix} 26 & 6 & 2 \\ -3 & 3 & -3 \\ 5 & 15 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} -50 & -6 & 7 \\ 11 & 25 & 1 \\ -9 & -34 & 51 \end{bmatrix}$$

(d) $$\left(AC + 3B^T\right)^T = \left( \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 5 \\ 1 & 6 & 0 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 5 & 0 \end{bmatrix} + 3 \begin{bmatrix} 1 & 4 \\ 3 & 0 \\ -2 & 1 \end{bmatrix} \right)^T$$

$$= \left( \begin{bmatrix} 9 & 4 \\ 24 & 1 \\ 8 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 12 \\ 9 & 0 \\ -6 & 3 \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} 12 & 33 & 2 \\ 16 & 1 & 3 \end{bmatrix}$$

(e) $$BCB = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 3 \\ 13 & 24 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -15 & 13 \\ 109 & 39 & -2 \end{bmatrix}$$

(f) $tr\left(\frac{1}{3}BAC\right) = \frac{1}{3}tr(BAC)$

$$= \frac{1}{3}tr\left(\begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 5 \\ 1 & 6 & 0 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 5 & 0 \end{bmatrix}\right)$$

$$= \frac{1}{3}tr\left(\begin{bmatrix} -1 & -13 & 16 \\ 5 & 14 & 4 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 5 & 0 \end{bmatrix}\right)$$

$$= \frac{1}{3}tr\left(\begin{bmatrix} 65 & 7 \\ 44 & 16 \end{bmatrix}\right)$$

$$= \frac{1}{3}(81) = 27$$

23. (a) $A+(B+C) = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \left( \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix} \right)$

$$= \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & -1 \\ -2 & 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 8 \\ 3 & 3 & -1 \\ -1 & 5 & 1 \end{bmatrix}$$

$$(A+B)+C = \left( \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \right) + \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 5 \\ 1 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 8 \\ 3 & 3 & -1 \\ -1 & 5 & 1 \end{bmatrix}$$

(b) $(AB)C = \left( \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 8 & -6 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 28 & -21 & 3 \\ 4 & -3 & 2 \\ 6 & -3 & 2 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 6 & -2 & 5 \\ 4 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 28 & -21 & 3 \\ 4 & -3 & 2 \\ 6 & -3 & 2 \end{bmatrix}$$

(c) $$A(B+C) = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & -1 \\ -2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 29 & 2 \\ 3 & 0 & -4 \\ -1 & 7 & 3 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 & -6 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} + \begin{bmatrix} -8 & 21 & 8 \\ 2 & -1 & -3 \\ -2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 29 & 2 \\ 3 & 0 & -4 \\ -1 & 7 & 3 \end{bmatrix}$$

(d) $$a(bC) = 2\left( 3 \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix} \right) = 2 \begin{bmatrix} 0 & 3 & 9 \\ 6 & 0 & 0 \\ -6 & 12 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 18 \\ 12 & 0 & 0 \\ -12 & 24 & 6 \end{bmatrix}$$

$$(ab)C = (2 \cdot 3)C = 6C = 6 \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 18 \\ 12 & 0 & 0 \\ -12 & 24 & 6 \end{bmatrix}$$

(e) $$(A+B)^T = \left( \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \right)^T = \begin{bmatrix} 2 & 2 & 5 \\ 1 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & -1 & 0 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & -1 & 0 \end{bmatrix}$$

(f) $$(AB)^T = \left( \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \right)^T = \begin{bmatrix} 3 & 8 & -6 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 & 1 \\ 8 & 1 & 2 \\ -6 & -1 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 8 & 1 & 2 \\ -6 & -1 & -1 \end{bmatrix}$$

24. $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 11 \\ 3 & 1 & 1 \end{bmatrix}$ , $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ , $B = \begin{bmatrix} 5 \\ 19 \\ 11 \end{bmatrix}$

25. $A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$ , $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ , $B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

26.
$$\begin{array}{rcrcrcl} -x & + & 7y & & & = & 0 \\ & + & 4y & + & 3z & = & 0 \\ 6x & & & - & 2z & = & 0 \end{array}$$

27.
$$\begin{array}{rcrcrcrcl} 1/3x_1 & + & 1/2x_2 & & & - & 1/6x_4 & = & 1/2 \\ 2/3x_1 & + & 1/6x_2 & - & 1/2x_3 & & & = & 1/4 \end{array}$$

28. (a) Add -5 times row 1 to row 2.
    (b) Not elementary.
    (c) Interchange rows 1 and 2.
    (d) Multiply row 2 by $-1/2$.
    (e) Not elementary.
    (f) Add 9 times row 2 to row 4.

29. The operation that transforms $A$ into $B$ is: add -2 times row 1 to row 3. Applying this same operation to I yields:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

30. The operation that transforms $B$ into $A$ is: add 2 times row 1 to row 3. Applying this same operation to I yields:

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

31. The operation that transforms $B$ into $C$ is: interchange rows 2 and 3. Applying this same operation to I yields:

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

32. The operation that transforms $C$ into $B$ is: interchange rows 2 and 3. Applying this same operation to I yields:

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

33.
$$\left[\begin{array}{cc:cc} 2 & 3 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc:cc} 1 & 3/2 & 1/2 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc:cc} 1 & 3/2 & 1/2 & 0 \\ 0 & -3/2 & -3/2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc:cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 5/3 & -2/3 \end{array}\right] \longrightarrow \left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 5/3 & -2/3 \end{array}\right]$$

Thus, the inverse is: $\begin{bmatrix} -2 & 1 \\ 5/3 & -2/3 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 2 & 1 & \cdot & 1 & 0 & 0 \\ 0 & 2 & 2 & \cdot & 0 & 1 & 0 \\ 0 & 0 & 4 & \cdot & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & \cdot & 1 & 0 & 0 \\ 0 & 1 & 1 & \cdot & 0 & 1/2 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 1/4 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 0 & -1 & \cdot & 1 & -1 & 0 \\ 0 & 1 & 1 & \cdot & 0 & 1/2 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 1/4 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & \cdot & 1 & -1 & 1/4 \\ 0 & 1 & 0 & \cdot & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 1/4 \end{bmatrix}$

Thus, the inverse is $\begin{bmatrix} 1 & -1 & 1/4 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/4 \end{bmatrix}$

35. $\begin{bmatrix} 1 & -5 & 3 & \cdot & 1 & 0 & 0 \\ 2 & 0 & 1 & \cdot & 0 & 1 & 0 \\ 1 & 1 & 1 & \cdot & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -5 & 3 & \cdot & 1 & 0 & 0 \\ 0 & 10 & -5 & \cdot & -2 & 1 & 0 \\ 0 & 6 & -2 & \cdot & -1 & 0 & 1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 0 & 1/2 & \cdot & 0 & 5/10 & 0 \\ 0 & 1 & -1/2 & \cdot & -2/10 & 1/10 & 0 \\ 0 & 0 & 1 & \cdot & 2/10 & -6/10 & 1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & \cdot & -1/10 & 8/10 & -5/10 \\ 0 & 1 & 0 & \cdot & -1/10 & -2/10 & 5/10 \\ 0 & 0 & 1 & \cdot & 2/10 & -6/10 & 10/10 \end{bmatrix}$

Thus, the inverse is: $\begin{bmatrix} -1/10 & 8/10 & -5/10 \\ -1/10 & -2/10 & 5/10 \\ 2/10 & -6/10 & 10/10 \end{bmatrix}$

36. $\begin{bmatrix} 1 & -5 & 5 & \cdot & 1 & 0 & 0 \\ 4 & 3 & 2 & \cdot & 0 & 1 & 0 \\ 9 & 1 & 9 & \cdot & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -5 & 5 & \cdot & 1 & 0 & 0 \\ 0 & 23 & -18 & \cdot & -4 & 1 & 0 \\ 0 & 46 & -36 & \cdot & -9 & 0 & 1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & -5 & 5 & \cdot & 1 & 0 & 0 \\ 0 & 1 & -18/23 & \cdot & -4/23 & 1/23 & 0 \\ 0 & 0 & 0 & \cdot & -1 & -2 & 1 \end{bmatrix}$

Since we have obtained a row of zeros on the left side, the matrix is not invertible.

37. $\begin{bmatrix} 1/2 & 0 & 0 & \cdot & 1 & 0 & 0 \\ 1/4 & 1/2 & 0 & \cdot & 0 & 1 & 0 \\ 1 & 1/4 & 1/2 & \cdot & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & \cdot & 2 & 0 & 0 \\ 0 & 1/2 & 0 & \cdot & -1/2 & 1 & 0 \\ 0 & 1/4 & 1/2 & \cdot & -2 & 0 & 1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & \cdot & 2 & 0 & 0 \\ 0 & 1 & 0 & \cdot & -1 & 2 & 0 \\ 0 & 0 & 1/2 & \cdot & -7/4 & -1/2 & 1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & \cdot & 2 & 0 & 0 \\ 0 & 1 & 0 & \cdot & -1 & 2 & 0 \\ 0 & 0 & 1 & \cdot & -7/2 & -1 & 2 \end{bmatrix}$

Thus, the inverse is: $\begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ -7/2 & -1 & 2 \end{bmatrix}$

38. $$\left[\begin{array}{cccc|cccc} 1 & -3 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & -3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6/5 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & -2/5 & 0 & 0 & -2/5 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -5/2 \end{array}\right]$$

Thus, the inverse is: $\begin{bmatrix} 4 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & -5/2 \end{bmatrix}$

39. $$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & k_1 & 0 & 1 & 0 \\ 1 & k_2 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & k_2 & 0 & 0 & 0 & 1 \\ 0 & 1 & k_1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -k_1k_2 & 0 & -k_2 & 1 \\ 0 & 1 & k_1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & k_1k_2 & -k_2 & 1 \\ 0 & 1 & 0 & -k_1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

Thus the inverse is: $\begin{bmatrix} k_1k_2 & -k_2 & 1 \\ -k_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

40. $$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ k & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & k & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -k & 1 & 0 & 0 \\ 0 & k & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & k^2 & -k & 1 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdot & -k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & k^2 & -k & 1 & 0 \\ 0 & 0 & 0 & 1 & \cdot & -k^3 & k^2 & -k & 1 \end{array}\right]$$

Thus, the inverse is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 \\ k^2 & -k & 1 & 0 \\ -k^3 & k^2 & -k & 1 \end{bmatrix}$

## SECTION 1.6 *FURTHER RESULTS ON SYSTEMS OF EQUATIONS AND INVERTIBILITY*

41. $\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ 32 \end{bmatrix}$ so $x = -9$ , $y = 32$

42. $\begin{bmatrix} 1 & 4 \\ 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{1}{-19} \begin{bmatrix} 1 & -4 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{-1}{19} \begin{bmatrix} 11 \\ -17 \end{bmatrix}$ so $x = -11/19$ , $y = 17/19$

43. $\left[\begin{array}{ccccccc} 1 & 6 & 5 & \cdot & 1 & 0 & 0 \\ 0 & 1 & 1 & \cdot & 0 & 1 & 0 \\ 0 & 0 & 2 & \cdot & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccccc} 1 & 0 & -1 & \cdot & 1 & -6 & 0 \\ 0 & 1 & 1 & \cdot & 0 & 1 & 0 \\ 0 & 0 & 2 & \cdot & 0 & 0 & 1 \end{array}\right]$

$\longrightarrow \left[\begin{array}{ccccccc} 1 & 0 & 0 & \cdot & 1 & -6 & 1/2 \\ 0 & 1 & 0 & \cdot & 0 & 1 & -1/2 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 1/2 \end{array}\right]$

$\begin{bmatrix} 1 & -6 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ -2 \end{bmatrix}$ so $x = 7$ , $y = 1$ , $z = -2$

44. $\left[\begin{array}{ccccccc} 1 & -1 & 2 & \cdot & 1 & 0 & 0 \\ -2 & 1 & 1 & \cdot & 0 & 1 & 0 \\ -3 & 1 & 5 & \cdot & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccccc} 1 & -1 & 2 & \cdot & 1 & 0 & 0 \\ 0 & -1 & 5 & \cdot & 2 & 1 & 0 \\ 0 & -2 & 11 & \cdot & 3 & 0 & 1 \end{array}\right]$

$\longrightarrow \left[\begin{array}{ccccccc} 1 & 0 & -3 & \cdot & -1 & -1 & 0 \\ 0 & 1 & -5 & \cdot & -2 & -1 & 0 \\ 0 & 0 & 1 & \cdot & -1 & -2 & 1 \end{array}\right]$

$\longrightarrow \left[\begin{array}{ccccccc} 1 & 0 & 0 & \cdot & -4 & -7 & 3 \\ 0 & 1 & 0 & \cdot & -7 & -11 & 5 \\ 0 & 0 & 1 & \cdot & -1 & -2 & 1 \end{array}\right]$

$\begin{bmatrix} -4 & -7 & 3 \\ -7 & -11 & 5 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ so $x_1 = 0$ , $x_2 = -1$ , $x_3 = 0$

45. $\left[\begin{array}{ccccccc} 2 & -3 & 10 & \cdot & 1 & 0 & 0 \\ 0 & -1 & 2 & \cdot & 0 & 1 & 0 \\ -4 & 8 & -19 & \cdot & 0 & 0 & 1 \end{array}\right]$

$$\longrightarrow \begin{bmatrix} 1 & -3/2 & 5 & \cdot & 1/2 & 0 & 0 \\ 0 & -1 & 2 & \cdot & 0 & 1 & 0 \\ 0 & 2 & 1 & \cdot & 2 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 2 & \cdot & 1/2 & -3/2 & 0 \\ 0 & 1 & -2 & \cdot & 0 & -1 & 0 \\ 0 & 0 & 5 & \cdot & 1 & 2 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & \cdot & -3/10 & -23/10 & -2/5 \\ 0 & 1 & 0 & \cdot & 4/5 & -1/5 & 2/5 \\ 0 & 0 & 1 & \cdot & 2/5 & 2/5 & 1/5 \end{bmatrix}$$

$$\begin{bmatrix} -3/10 & -23/10 & -2/5 \\ 4/5 & -1/5 & 2/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 10 \\ 30 \\ -20 \end{bmatrix} = \begin{bmatrix} -64 \\ -6 \\ 12 \end{bmatrix} \quad \text{so} \quad x_1 = -64 \;,\quad x_2 = -6 \;,\quad x_3 = 12$$

46. $$\begin{bmatrix} 1 & -2 & -6 & -2 & \cdot & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & \cdot & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdot & 0 & 0 & 1 & 0 \\ 3 & 3 & 7 & 4 & \cdot & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -6 & -2 & \cdot & 1 & 0 & 0 & 0 \\ 0 & 3 & 8 & 3 & \cdot & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & 2 & \cdot & -1 & 0 & 1 & 0 \\ 0 & 9 & 25 & 10 & \cdot & -3 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -2/3 & 0 & \cdot & 1/3 & 2/3 & 0 & 0 \\ 0 & 1 & 8/3 & 1 & \cdot & -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & -1/3 & 0 & \cdot & -1/3 & -2/3 & 1 & 0 \\ 0 & 0 & 1 & 1 & \cdot & 0 & -3 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 1 & 2 & -2 & 0 \\ 0 & 1 & 0 & 1 & \cdot & -3 & -5 & 8 & 0 \\ 0 & 0 & 1 & 0 & \cdot & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 & \cdot & -1 & -5 & 3 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 1 & 2 & -2 & 0 \\ 0 & 1 & 0 & 0 & \cdot & -2 & 0 & 5 & -1 \\ 0 & 0 & 1 & 0 & \cdot & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 & \cdot & -1 & -5 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -2 & 0 \\ -2 & 0 & 5 & -1 \\ 1 & 2 & -3 & 0 \\ -1 & -5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ -3 \\ 12 \end{bmatrix} \quad \text{so} \quad x_1 = -2 \;,\quad x_2 = -6 \;,\quad x_3 = -3 \;,\quad x_4 = 12$$

47. $$\begin{bmatrix} .3 & .4 \\ .7 & .6 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{-.1} \begin{bmatrix} .6 & -.4 \\ -.7 & .3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 7 & -3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -6b_1 + 4b_2 \\ 7b_1 - 3b_2 \end{bmatrix}$$

so $x = -6b_1 + 4b_2$ and $y = 7b_1 - 3b_2$.

48. $$\begin{bmatrix} 1 & 1 & 1 & \cdot & 1 & 0 & 0 \\ 3 & 4 & 5 & \cdot & 0 & 1 & 0 \\ -2 & 0 & 1 & \cdot & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & \cdot & 1 & 0 & 0 \\ 0 & 1 & 2 & \cdot & -3 & 1 & 0 \\ 0 & 2 & 3 & \cdot & 2 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 & \cdot & 4 & -1 & 0 \\ 0 & 1 & 2 & \cdot & -3 & 1 & 0 \\ 0 & 0 & -1 & \cdot & 8 & -2 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & \cdot & -4 & 1 & -1 \\ 0 & 1 & 0 & \cdot & 13 & -3 & 2 \\ 0 & 0 & 1 & \cdot & -8 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 1 & -1 \\ 13 & -3 & 2 \\ -8 & 2 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -4b_1 + b_2 - b_3 \\ 13b_1 - 3b_2 + 2b_3 \\ -8b_1 + 2b_2 - b_3 \end{bmatrix}$$

so $x_1 = -4b_1 + b_2 - b_3$ , $x_2 = 13b_1 - 3b_2 + 2b_3$ , $x_3 = -8b_1 + 2b_2 - b_3$.

49. $\begin{bmatrix} 6 & -3 & b_1 \\ -10 & 5 & b_2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/2 & b_1/6 \\ 0 & 0 & b_2 + 10b_1/6 \end{bmatrix}$

Thus, the only condition required for consistency is:

$$b_2 + \tfrac{10}{6} b_1 = 0$$

i.e., $b_2 + \frac{5}{3} b_1 = 0$

$$b_2 = -\tfrac{5}{3} b_1$$

50. $\begin{bmatrix} 1 & 4 & -3 & b_1 \\ 2 & 6 & 5 & b_2 \\ 1 & 6 & -14 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & -3 & b_1 \\ 0 & -2 & 11 & b_2 - 2b_1 \\ 0 & 2 & -11 & b_3 - b_1 \end{bmatrix}$

$$\longrightarrow \begin{bmatrix} 1 & 4 & -3 & b_1 \\ 0 & 1 & -11/2 & -1/2(b_2 - 2b_1) \\ 0 & 0 & 0 & (b_3 - b_1) + (b_2 - 2b_1) \end{bmatrix}$$

Thus, the only condition required for consistency is:

$$(b_3 - b_1) + (b_2 - 2b_1) = 0$$

i.e., $b_3 + b_2 - b_1 = 0$

$$b_3 = b_1 - b_2$$

51. $\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 2 & 7 & 3 & b_2 \\ -1 & -1 & 2 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & 1 & 1 & b_2 - 2b_1 \\ 0 & 2 & 3 & b_3 + b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & 1 & 1 & b_2 - 2b_1 \\ 0 & 0 & 1 & (b_3 + b_1) - 2(b_2 - 2b_1) \end{bmatrix}$

Thus, there are no conditions required for consistency.

52. $\begin{bmatrix} 1 & 4 & 7 & b_1 \\ 2 & 5 & 1 & b_2 \\ 1 & 7 & 20 & b_3 \\ -1 & 2 & 19 & b_4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 7 & b_1 \\ 0 & -3 & -13 & b_2 - 2b_1 \\ 0 & 3 & 13 & b_3 - b_1 \\ 0 & 6 & 26 & b_4 + b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 7 & b_1 \\ 0 & 1 & 13/3 & -1/3(b_2 - 2b_1) \\ 0 & 0 & 0 & (b_3 - b_1) + (b_2 - 2b_1) \\ 0 & 0 & 0 & (b_4 + b_1) + 2(b_2 - 2b_1) \end{bmatrix}$

Thus, the only conditions required for consistency are:

$$(b_3 - b_1) + (b_2 - 2b_1) = 0 \quad \text{and} \quad (b_4 + b_1) + 2(b_2 - 2b_1) = 0$$

$$\text{i.e.,} \quad b_3 + b_2 - 3b_1 = 0 \quad \text{and} \quad b_4 + 2b_2 - 3b_1 = 0$$

$$b_3 = 3b_1 - b_2 \quad \text{and} \quad b_4 = 3b_1 - 2b_2$$